J. Symbolic Computation (1989) 8, 305–308

Symbolic Computation of Hyperbolicity Regions for Systems of Two-Phase Flow Conservation Laws Using Maple

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(*Received* 21 *October* 1988)

Symbolic algebra packages have recently been making a large impact on both pure and applied mathematical research. In this study, an application of the symbolic manipulation system MAPLE to the mathematical modelling of multiphase flows is discussed. The physical modelling problem is an extremely hard one, and there is little consensus on the correct form of the governing equations for such flows. Moreover, many such sets of equations are ill-posed in the sense that they admit complex eigenvalues (wave speeds). A procedure has been written in MAPLE command language to examine a set of partial differential equations to determine their character (elliptic, hyperbolic or parabolic). Previously, for many systems this computation could take so long that it was unrealistic to attempt. Now however such calculations may be made so quickly that it is possible to undertake parametric studies of the equations in order to improve basic modelling techniques.

1. Introduction

This paper gives details of symbolic calculations which have been carried out using MAPLE to study the mathematical structure and well-posedness of mathematical models for multiphase flows. Typically, a multiphase flow involves the motion of a number of different substances relative to each other. Common scenarios which have been studied include: gas bubbles in fluids, solid particles in gases and mixtures of gas, oil and water flowing in porous media. Owing to the mathematical impracticability of tracking all the phase boundaries of different scattered components of the mixture, ensemble averaging is commonly employed to give a continuum description of the flow. Because of this, it is frequently the case that submodels must be formulated in order to re-insert some of the important physics which the averaging has removed. The content of such models is currently a subject of extremely diverse opinion, and agreement on the correct modelling is by no means unanimous. In the next section we will explain some of the unfortunate mathematical consequences of incorrect modelling. The relevant fact, however, is that it is crucial to determine certain mathematical properties of the system of conservation laws which result from the mathematical modelling process. This may involve the symbolic manipulation of large matrices with complicated terms, consideration of real and complex polynomials, the rearrangement of lengthy partial differential equations, and also integration and differentiation.

One of the general advantages of symbolic computation systems such as MAPLE is that they have the ability to save time and provide extra convenience for mathematicians on a day-to-day basis. The present application, however, has shown the capability of MAPLE to accomplish much more than this, as it has allowed computations which previously may

0747–7171/89/090305+04 $03.00/0

have taken many weeks to be carried out quickly and without error. This has made parametric studies of such sets of conservation laws feasible, which were simply impossible a few years ago. Indeed, it is not an unrealistic claim to suggest that such parametric studies have changed the approach to the mathematical modelling of multiphase flow systems as it is now possible to calculate almost instantly the effect of a change in a submodel or constitutive law. Previously, the work involved in performing the necessary calculations for some submodels was so great that they could not realistically be studied. One approach which was formerly used by the author in such cases was to study the resulting matrices numerically, but unfortunately such matrices and the systems of linear equations associated with them are frequently ill-conditioned. This makes the present algorithm all the more welcome. It should also be mentioned that MAPLE was used to write the algorithm as it incorporates a convenient plotting facility. This is important, as it is often instructive to draw pictures of complicated regions of phase space.

As well as two-phase flow models, the program has been used to study other systems of conservation laws. Some of these represent mathematical models of other physical problems (see, for example, Bell, Trangenstein & Schubin (1986)), whilst others are prototype problems for the study of mixed hyperbolic–elliptic systems (see, for example Holden (1987)). Such prototype systems can be constructed especially easily in this fashion and forced to have virtually any desired property.

2. Mathematical Considerations

We consider here functions of a single spatial coordinate x and time t. (Extensions to higher dimensions are not difficult as far as the algorithm is concerned, but interpretation of the results more difficult.) Typically the systems we have considered involve variables such as the density ρ, internal energy e, pressure p, number density N, velocity u and cross-sectional area A $(A_1+A_2 = A)$. Numerical subscripts denote the phase to which each variable refers, and other subscripts denote differentiation. The physical processes are modelled by a system of conservation laws of the form

$$\mathbf{w}_t+[\mathbf{f}(\mathbf{w})]_x = \mathbf{h},$$

or more generally

$$A(\mathbf{w}, x, t)\mathbf{w}_t+B(\mathbf{w}, x, t)\mathbf{w}_x = \mathbf{h}(\mathbf{w}, x, t), \tag{1}$$

where $\mathbf{w}$ is a vector of N unknowns, A and B are $N\times N$ matrices and $\mathbf{h}$ is an N-vector of source terms. In fact $\mathbf{h}$ is irrelevant to the analysis which will be performed, and so will be set to zero henceforth without loss of generality. Without going into all of the mathematical details (for fuller information see, for example Smoller (1983)) we expect that the "sound speeds" of the system, given by values λ such that $\det(B-\lambda A) = 0$, will all be real for a well-posed system, where small changes in the initial or boundary data lead only to small changes in the solution. When all of the eigenvalues λ are real, we can identify the Pfaffian ordinary differential equations which hold along the characteristic curves, and possibly integrate them to retrieve the Riemann invariants of the system. If some of the λ's are complex, however, the implication is that the problem is ill-posed and a boundary condition at $t = \infty$ is required, violating causality. A subsidiary consequence of complex eigenvalues is that it can be easily shown that they render any numerical finite-difference method prone to numerical instability. Ostensibly, therefore, the problem is a simple one, involving only the examination of a determinant. In practice, however, quite apart from the fact that the relevant determinant is likely to be awkward and lengthy to calculate by hand, the conservation laws are often not in the convenient form (1) and considerable work must be carried out on them to achieve the correct form. Also, the resulting polynomial equation for the eigenvalues λ is, typically, extremely long and complicated, and it is therefore difficult to determine the regions in phase space (if any) in which all the eigenvalues of the system are real so that the system is totally hyperbolic.

3. Algorithm Details and Illustrative Results

Having given the mathematical details of the problem, some comments on the algorithm are relevant. It is based around the MAPLE "deter" function to find the determinant of a matrix. Factoring, integration and further calculations are then driven by prompts from the user. Before the required determinants and discriminants can be calculated, however, some work must be carried out on the equations themselves to ensure that they are in the correct form. When the process is carried out by hand it is frequently this part of the work which takes as long as anything else. Input can either be from a file which contains the conservation laws and primitive variables, or from the terminal. The algorithm is not long (about 200 full lines of condensed MAPLE commands) and the author will be glad to supply copies. Execution is carried out by MAPLE running on a VAX 8700 in about 1 min CPU time for an 8×8 full system of equations. As an example of a few of the results which have been calculated using the program, consider the following system of incompressible two-phase flow equations which has frequently (and erroneously) been used for numerical calculations

$$(\rho_1 A_1)_t + (\rho_1 A_1 u_1)_x = 0,$$
$$(\rho_2 A_2)_t + (\rho_2 A_2 u_2)_x = 0,$$
$$(\rho_1 A_1 u_1)_t + (\rho_1 A_1 u_1^2)_x + A_1 p_x = 0,$$
$$(\rho_2 A_2 u_2)_t + (\rho_2 A_2 u_2^2)_x + A_2 p_x = 0.$$

The following results were obtained using the algorithm:

$$c(\lambda) = -A_1\rho_1(-A+A_1)\rho_2(A_1\rho_1 u_1^2 - A\rho_1 u_1^2 - 2A_1\lambda\rho_1 u_1 + 2A\lambda\rho_1 u_1 - A_1\rho_2 u_2^2 + 2A_1\rho_2\lambda u_2 + A_1\lambda^2\rho_1 - A\lambda^2\rho_1 - A_1\rho_2\lambda^2),$$
$$\Omega = 4(u_1-u_2)^2(-A+A_1)^3\rho_2^3\rho_1^3 A_1^3$$

Here $c(\lambda)$ represents the characteristic polynomial of the system and Ω the discriminant of $c(\lambda)$. In this case the determinant shows that two of the eigenvalues are zero, which corresponds to the incompressibility of both phases. The remaining eigenvalues satisfy a quadratic equation, whose discriminant is negative unless $u_1 = u_2$ by virtue of the fact that $-A + A_1 = -A_2$. Thus, we have shown that under all circumstances except trivial ones the system contains two real and two complex complex eigenvalues and is ill-posed.

Finally, we consider a slightly more realistic example. The system above is replaced by

$$(\rho_1 A_1)_t + (\rho_1 A_1 u_1)_x = 0,$$
$$(\rho_2 A_2)_t + (\rho_2 A_2 u_2)_x = 0,$$
$$(\rho_1 A_1 u_1)_t + (\rho_1 A_1 u_1^2)_x + A_1 p_x = 0,$$
$$(\rho_2 A_2 u_2)_t + (\rho_2 A_2 u_2^2)_x + A_2 p_x = 0,$$

$$[\rho_1 A_1(u_1^2/2+e_1)]_t+[\rho_1 A_1 u_1(u_1^2/2+e_1+p/\rho_1)]_x+p(A_2 u_2)_x=0,$$

$$(N_2)_t+(u_2 N_2)_x=0,$$

so that phase 1 is assumed compressible (obeying the perfect gas law), and a number density equation is added for phase 2. This has been considered as a model for compressible liquid/gas flow. Now the results become

$$\begin{aligned} c(\lambda) &= (u_2-\lambda)(u_1-\lambda)(u_2^2u_1^2\rho_1\rho_2-2u_2^2u_1\lambda\rho_1\rho_2+u_2^2\lambda^2\rho_1\rho_2-u_2^2\rho_1\rho_2c^2 \\ &\quad -2u_2\lambda u_1^2\rho_1\rho_2+4u_2u_1\lambda^2\rho_1\rho_2-2u_2\lambda^3\rho_1\rho_2+2u_2\lambda c^2\rho_1\rho_2+\lambda^2u_1^2\rho_1\rho_2-2u_1\lambda^3\rho_1\rho_2 \\ &\quad +\lambda^4\rho_1\rho_2-\lambda^2c^2\rho_1\rho_2+2\lambda\rho_1^2u_1c^2A_2/A_1-\rho_1^2\lambda^3c^2A_2/A_1-\rho_1^2u_1^2c^2A_2/A_1)/(\rho_1\rho_2) \\ \Omega &= -16Vq+48Vq(V-q)-48qV^3-336V^2q^2-48q^3V+16V^4q \\ &\quad -48V^2q^2(V+q)-16q^4V \end{aligned}$$

where $V=(u_2-u_1)^2/c^2$, $q=(\rho_1A_2)/(\rho_2A_1)$. It is then an easy mater to rearrange this using MAPLE to show that the system has 4 real and 2 complex eigenvalues whenever $V<(1+q^{1/3})^3$ and 6 real eigenvalues otherwise. Of course, space has only permitted us to exhibit two of the simpler calculations which we have made using the algorithm, but many much more complicated cases have been examined to reveal new results concerning the well-posedness of two-phase flow conservation laws.

In conclusion, the algorithm which has been developed constitutes an important research tool for the study of two-phase flow. It has already been used extensively and will continue to be expanded and developed.

Thanks are due to Professor Don Drew of RPI, New York for the original motivation to study mixed systems of conservation laws using symbolic algebra programs.

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