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A REVIEW OF LINEAR AND NONLINEAR CAUCHY SINGULAR INTEGRAL AND INTEGRO-DIFFERENTIAL EQUATIONS ARISING IN MECHANICS

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ABSTRACT. This study is primarily concerned with the presentation of a review of a collection (which could be regarded as a "test set") of linear and nonlinear singular integro-differential equations with Cauchy kernels, all of which arise from practical applications in Applied Mathematics and Mathematical Physics. The main objective of this review is to provide numerical analysts and researchers interested in algorithm development with model problems of genuine scientific interest on which to test their algorithms.

Brief details of the methodology of derivation of the equations are provided and, where possible, existence, uniqueness and asymptotic results are discussed. References are also given to other studies that have dealt with similar problems. The importance of carrying out the necessary mathematical analysis is emphasized for one class of problems where it is shown that the solution abruptly ceases to exist as a parameter is varied. It is further shown that developing asymptotic estimates for the behavior of the solutions is very often a crucial component in the design of effective numerical methods. The importance of regularization is discussed for a class of problems, specific conclusions are drawn and recommendations are discussed. An appendix contains further related problems that may be used for further comparison purposes.

1. Introduction. Many practical problems in elasticity, crack theory, wing theory and fluid flow give rise to singular integral equations with Cauchy kernels (see, for example [**3, 37, 61**]). For linear equations, a great deal of theory exists. Indeed, Muskhelishvili in his famous book [**77**] presents a thorough analysis of linear singular integral equations (LSIEs) providing closed form solutions. For this class of problem a considerable number of numerical techniques also exist. Schemes based on Galerkin methods [**39, 44**], collocation methods [**14, 16, 17,**

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20] and quadrature methods [**25, 64**] have proved successful and efficient; in all cases the necessary convergence theory is reasonably well developed.

For singular integral and integro-differential equations with Cauchy kernels that are nonlinear (referred to henceforth as NLSIEs and NLSIDEs respectively), matters are very different, and little has been published. This is undoubtedly because the theoretical difficulties posed by such equations are immense. No general well-posedeness results appear to exist; closed form analytic solutions for particular problems are very much the exception rather than the rule; and smoothness results and asymptotic expressions are not always easy to obtain. Consequently, such equations have also received little attention from numerical analysts and algorithm designers.

The main goal of this study is to review mechanics problems that have led to (in particular nonlinear) equations with Cauchy kernels. The results may be regarded as a collection of model problems (all of which come from practical problems, and are therefore of scientific, as well as theoretical interest) that might serve as a tool for researchers interested in the development and analysis of numerical methods for NLSIEs and NLSIDEs. Some illustrative examples of successful numerical methods will also be presented.

Other examples of collections of test problems have proved to be important to the numerical analysis community in the development of efficient and accurate algorithms. Perhaps the best known are the Harwell-Boeing sparse matrix test problem collection [**19**] and the netlib collection of linear programming problems [**34**]. The test sets of Hock [**42**] and Moré [**75**] provide well-known examples of a collection of nonlinear problems for nonlinear programming and unconstrained optimization. As far as the numerical solution of ordinary differential equations (ODE's) is concerned, test problems made a great impact on the development of robust code for their solution. Two of the most famous test problem collections for ODE's are the 75 test problems in Hull et al. [**43**] and the test problems in Krogh [**65**]; it is no exaggeration to say that one or both of these problem sets have been used to test almost every ODE code written since their appearance.

Hull's and Krogh's collections are designed to test particular aspects of ODE codes and contain so-called “toy problems” as well as realistic problems such as the five-body problem. The collection presented in this paper is different in that it is formed only from problems that arise in application areas.

The development and analysis of numerical solutions for the problems that appear in this collection is an important challenge to all numerical analysts. We follow the style of Moré's presentation [**74**] in that we provide details of the origin and the problem that each equation purports to model. Although we do not fully derive every equation, each is posed in its most appropriate form, complete with boundary conditions, typical parameter sizes (where available), any existing analytical results such as asymptotic expansions and, where appropriate, details of techniques that have previously been employed for its numerical solution.

As we shall demonstrate, the different properties of different types of equation can have a large influence on how best to solve them. It will also become clear that, in many cases, successful numerical solution of such equations is not possible unless some theoretical work is carried out as well. In particular, it is often crucial to determine the asymptotic behavior of the solution at various key points in the solution domain; in addition to helping with numerical aspects of the problem, asymptotic analysis of this sort can also influence one's choice of regularization (where appropriate) and warn of potential difficulties that might be inherent in the problem. The crucial dependence of one's choice of numerical scheme on the asymptotic details of the solution also indicates that it is extremely difficult to design widely applicable methods for NLSIDEs; often an ad hoc approach is the only way forward.

The ten problems presented below are arranged roughly in order of increasing numerical difficulty, though for such problems relative difficulty must, to some extent, be a matter of opinion. The collection also performs the dual function of indicating what sort of problems one might be able to solve in closed form and the current state of theory regarding existence and uniqueness for such equations.

2. Previous work.

2.1 *Linear singular integral and integro-differential equations.* In recent years, the theory as well as the numerical solution of linear Cauchy

SIEs has attracted the attention of a number of researchers, and considerable advances have been achieved in those areas. From a theoretical point of view, the books by Muskhelishvili [**77**], Gakhov [**33**] and Kravchenko and Litvinchuk [**63**] present a thorough analysis of the linear problem, providing closed form solutions. With regard to numerical solutions, a number of numerical techniques exist based variously on Galerkin methods, collocation methods, quadrature methods and spline approximation methods. Many papers study Galerkin methods for linear SIEs with constant coefficients [**38, 39, 41, 44**] as well as with variable coefficients [**22**]. Convergence in the L_2 norm is obtained. Superconvergence results are also available for the Galerkin method for calculating linear functionals of the solutions of a Cauchy SIE, see [**40**]. The collocation method was extensively studied by Elliott in [**20, 23**] for the case of a general linear SIE of the form (1) below, where the function $b(x)$ is such that there is a Hölder continuous function $q(x) > 0$ with $B(x) = q(x)b(x)$ a polynomial of degree m. Convergence rates were obtained in both the L_2 and uniform norms. Elliott's methods were based on collocation using the zeros of a class of orthogonal polynomials derived from the coefficients a and b. Collocation using the zeros of the first kind Chebyshev polynomial was investigated in [**8**] and further in [**16**], where convergence in the uniform norm was proved and convergence rates given. Collocation on the Chebyshev nodes simplifies the numerical method but the zeros of the orthogonal polynomial associated with the SIE are sometimes very difficult to calculate. Quadrature methods have been proposed by [**26, 46, 90**] to solve the general variable coefficient LSIE (1). The analysis of convergence of collocation-quadrature methods is presented for instance in [**53**] (see also [**47**]). More recently Berthold and co-workers [**5**] devised a fast algorithm for the quadrature method which permits the solution of the linear system arising from the method in $O(n \log n)$ operations. The uniform convergence of this algorithm was presented in [**54**]. Spline approximation methods were studied in [**24, 35, 36, 73**]. The approximation of SIEs using splines does not seem to be very popular, perhaps because the resulting integrals are very difficult to calculate exactly and hard to evaluate numerically. The convergence analysis is even harder and only a few special cases seem to have been discussed in the literature.

For many of the cases discussed above the convergence theory is reasonably well developed. Junghanns and Silbermann [**59**] presented a modern mathematical account of the above-mentioned numerical techniques together with their convergence analysis, for the case of the linear Cauchy integral equation

$$a(x)\phi(x) + \frac{1}{\pi} \fint_{-1}^{1} \frac{b(x)}{t-x}\,\phi(t)\,dt + \int_{-1}^{1} h(x,t)\phi(t)\,dt = f(x), \qquad x \in [-1,1], \tag{1}$$

where a, b, f and h are known functions and ϕ is unknown (see also more recent work, e.g., [**55, 60**]). Chakrabarti and Vanden Berghe [**11**] introduce an approximate method by expanding the unknown solution as a sum of singularly weighted well-behaved functions. Most recently Manam [**71**] produced a complete analytic solution to a particular singular integral equation involving a logarithmic as well as a Cauchy-type singularity.

Considerably fewer papers deal with either the theory or the numerical solution of LSIDEs. It appears that, for this class of equations, research has been concentrated on special cases, like Prandtl's equation for which there are some papers see, for instance, [**9, 10, 45, 72**]. Another class of linear singular integro-differential equations is studied in [**31**]. Spline collocation methods for LSIDEs were considered in [**85**].

2.2 *Nonlinear singular integral and integro-differential equations.* As has already been pointed out, not a great deal has appeared for either NLSIEs or NLSIDEs.

One type of NLSIE that has previously been studied in some detail is the Nekrasov equation ([**97**]). A number of studies have derived models that reduce to such equations; in particular, a model for ploughing free surface flows analyzed in [**93**] solved the governing NLSIE using collocation to reduce the problem to a system of nonlinear transcendental equations which were then solved using the Powell hybrid method [**83**] with an exponential grid. The Nekrasov equation was also solved numerically in [**12**] using high-order quadrature rules; the existence of positive numerical solutions was also proved.

An NLSIE arising from a formulation of the problem of flow seepage through a dam was studied in [**50**]. Collocation and Gaussian quadrature were employed to solve an equivalent equation numerically using

Newton's method for the case where the dam shape function was piecewise smooth, and in this case it was shown in [**49**] that convergence analysis of the numerical method was possible (the computational aspects of the algorithm were further investigated in [**51**]). Other nonlinear integral equation studies by Junghanns and his co-workers include [**56, 58**]. Most recently this author studied (see [**52**]) the optimal control of a parameterized family of nonlinear Cauchy integral equations. Collocation methods for NLSIEs were also investigated in [**56, 57**] where the results of [**60**] were used to generate numerical schemes.

Amer (in [**1**], see also [**2**]) produced sufficient conditions for the convergence of the modified Newton-Kantorovich method for the solution of a certain class of nonlinear singular integral equations.

Even less has appeared on NLSIDEs. Ladopoulos (see, for example [**66**] where there are also other references to relevant work) has developed new collocation-approximation methods for NLSIDEs in a Banach space setting; he has been able to demonstrate the existence of solutions to the resultant nonlinear system. This and related work may be found in [**67**]. Another author to tackle NLSIDEs is Wolfersdorf [**98**].

3. Problem review. Our problem collection ("test set") is presented below. Each of the problems in the set possesses distinctive characteristics and poses different numerical, asymptotic and analytical challenges. Any singular integro-differential equation solver that performs well on all of the test set problems will indeed be a powerful tool. Details of some other related problems that might also be used for testing purposes are given in Appendix 1.

P1 *Stewartson's lifting line equation.* Our first problem concerns Prandtl's "lifting line" model for a wing. If a semi-infinite wing of constant chord is replaced by a straight line L parallel to the wing leading edge, many of the key features of the flow may be captured by assuming that the circulation Γ is a function only of the distance y along L. By Kelvin's theorem, a trailing vortex sheet must exist behind L; using standard potential flow arguments (see, for example [**84**]) the strength of the trailing vortex may be determined in terms of Γ. An application of the Joukowski hypothesis then yields a singular integro-differential equation for $\Gamma(y)$. After recasting the problem slightly to replace the unknown function $\Gamma(y)$ by $S(x)$, the "lifting line" equation was derived in [**89**] in the form

$$\frac{1}{\pi}\,⨍_0^\infty \frac{S'(t)}{t-x}\,dt = S(x), \quad x>0 \tag{2}$$

subject to the boundary conditions $S(0)=1$, $S(\infty)=0$. This first order constant-coefficient equation (which also occurs naturally in the theory of elasticity: see for example [**61**, Section 32]) is posed over a semi-infinite range, and successful numerical methods must therefore deal correctly with "contributions from infinity." Alternatively, the equation may be transformed to a finite range, though in general this will introduce non-constant coefficients.

The equation is linear, but serves as a good test case for nonlinear codes as it provides one of the very few instances where practically all of the important properties of the solution may be determined. Stewartson [**89**] obtained asymptotic expansions for small and large x in the form

$$\begin{aligned} S(x) &\sim 1 - 2\left(\frac{x}{\pi}\right)^{1/2} - \frac{4}{3}\left(\frac{x}{\pi}\right)^{3/2}\left(\log 4x + \gamma - \frac{11}{3}\right) \\ &\quad + O_\rho\left(\frac{x}{\pi}\right)^{5/2}, \quad \text{as} \quad x\to 0 \\ S(x) &\sim \frac{1}{\pi x} + \frac{1}{(\pi x)^2}\,(\gamma + \log x) + \frac{1}{(\pi x)^3}\left(2(\log x)^2 + 4\gamma\log x - 1 - \frac{4\pi^2}{3}\right) \\ &\quad + \cdots, \quad \text{as} \quad x\to\infty. \end{aligned}$$

(Note that here O_ρ means that the order includes an unspecified power of $\log x$, and $\gamma = 0.5772\ldots$, is Euler's constant.) Using the Wiener-Hopf method (the solution may also be determined using the methods developed in [**95**]; see the Appendix for further details), Stewartson also found the closed form solution to (2), which is

$$S(x) = \frac{1}{\pi}\int_0^\infty \frac{e^{-tx}}{(1+t^2)^{3/4}}\,\exp\left[-\frac{1}{\pi}\int_0^t \frac{\log\theta}{1+\theta^2}\,d\theta\right]\,dt.$$

Numerical values of this solution are displayed in Table 1.

TABLE 1. Numerical values of solution to the lifting line equation.

x	0.0	0.2	0.4	0.6	0.8	1.0	1.2
$S(x)$	1.0	0.564	0.438	0.361	0.308	0.267	0.237

x	1.4	1.6	1.8	2.0	3.0	4.0
$S(x)$	0.212	0.192	0.175	0.161	0.113	0.086

Some further discussion of P1 is warranted. We note that, though (2) is a first order singular integro-differential equation, it requires two boundary conditions. This feature is common to equations of this kind, and it is worth examining a little further. To simplify matters, consider the much more tractable first order equation

$$\frac{1}{\pi} \fint_0^1 \frac{S'(t)}{t-x}\, dt = 1.$$

Inverting this equation using standard theory (see, for example [**77**]), we find that

$$S'(x) = -\sqrt{\frac{1-x}{x}} + \frac{C}{\sqrt{x(1-x)}}$$

and thus

$$S(x) = -\sqrt{x(1-x)} + (C - 1/2)\arcsin(2x-1) + D$$

where C and D are arbitrary constants. A number of choices are now possible: we could choose to specify $S(0)$ and $S(1)$, in which case we find that $C = 1/2 + (S(1) - S(0))/\pi$ and $D = (S(0) + S(1))/2$. In this case, however, for almost all choices of $S(0)$ and $S(1)$ the solution will have infinite gradient at both $x = 0$ and $x = 1$. Alternatively, we might choose C so that the gradient of S is zero at either $x = 0$ or $x = 1$ and specify either $S(0)$ or $S(1)$. Many other cases may easily be examined and enumerated, but the general message is that an nth order singular integro-differential equation normally requires $n+1$ boundary conditions, one of which may often usefully be regarded as a regularity condition.

Note that, for Cauchy integral equations, the notions of "order" and "required number of boundary conditions" are intimately linked to the related notion of "index". In this study, when we describe an equation as being "of nth order", we simply mean that the highest derivative (whether appearing under the integral sign or not) has order n. The statement above should therefore be regarded merely as a "helpful first iteration" in deciding how many boundary conditions are required. For more complicated equations and kernels, only a detailed examination of the index can guide us to the proper specification of boundary data (for more details see [**33**]).

P2 *Thwaites' sail equation.* Thwaites [**91**] considered the flow of an inviscid incompressible fluid past a two-dimensional flexible inelastic membrane (see also the earlier study by Voelz [**96**]). By assuming that the sail deflections were small, and carrying out a force balance on a sail element, Thwaites expressed the unknown vortex distribution strength that corresponds to the disturbance produced by the sail to a unidirectional potential flow in terms of the pressure difference across the sail. This allowed the problem to be posed as a single nondimensional equation for the deflection $S(x)$ of the sail in the form

$$\frac{1}{\pi} \fint_0^1 \frac{S''(t)}{t-x}\, dt = \lambda(\alpha - S'(x)) \tag{3}$$

with boundary conditions $S(0) = S(1) = 0$ and $S''(1) = 0$. Here α denotes the (small) angle between the sail and the wind and the key nondimensional parameter is

$$\lambda = \frac{2\rho U^2 c}{T}$$

where ρ denotes the density of the free stream, T (N/m^2) is the sail tension per unit length, U is the speed of the oncoming free stream and c is the sail trailing edge position. As in P1, we note that this second order equation requires two boundary conditions and a "regularity condition" which in this case is the Kutta condition $S''(1) = 0$; this ensures that the flow is smooth at the trailing edge of the sail.

Although the Thwaites sail equation is linear, it is second order and is posed over a finite range. For this reason, no exact solutions are known for nonzero α (the method of [**95**] is only applicable to constant-coefficient equations posed over a semi-infinite range, and attempts to transform P2 to a semi-infinite range inevitably introduce nonconstant coefficients). Some approximate solutions suitable for numerical verification purposes may be determined in asymptotic cases such as the limit of large tension ($\lambda = 0$) when the leading-order solution is $S(x) = 0$ and a regular perturbation in λ yields useful results. In some sense the equation also resembles an eigenvalue problem, for it can be shown that some solutions also satisfy $S''(0) = 0$. In these cases, the flow is smooth at the leading edge of the sail as well as at the trailing edge, and the lift on such sails is thus zero. A further test for numerical schemes lies in the fact that it soon transpires that for a given α, solutions may be possible for more than one value of λ.

It should be pointed out that Thwaites [**91**] did not pose the problem in quite the concise form of (3) but chose to transform the problem to

$$\begin{aligned} (4) \quad \Psi(\theta) + \lambda \Bigg\{ -\frac{1}{2\pi}\int_0^\theta \sin^2\xi \,⨍_0^\pi \frac{\Psi(\phi)\,d\phi}{\cos\xi - \cos\phi}\,d\xi \\ + \frac{1}{4\pi}\int_0^\pi \sin\varphi \int_0^\varphi \sin^2(\xi) ⨍_0^\pi \frac{\Psi(\phi)d\phi}{\cos\xi - \cos\phi}\,d\xi\,d\varphi \Bigg\} \\ + \lambda\left(\frac{3\pi}{8} - \frac{\theta}{2} - \frac{\sin\theta}{2}\right)\left[\frac{1}{\pi}\int_0^\pi \Psi(\theta)\,d\theta\right] = \lambda\left(\frac{3\pi}{8} - \frac{\theta}{2} - \frac{\sin\theta}{2}\right) \end{aligned}$$

where the parameter α has been absorbed through the transformation $S'(x) = \alpha\Psi(x)$. Thwaites then proceeded to solve (4) using an approximate method; though this gave acceptable results (3) may also be solved directly using a range of techniques.

P3 *Steady flag equation.* A two-dimensional sail under zero tension with one end free and with nonzero bending stiffness may be regarded as a flag or pennant. By performing a force balance in a similar manner to that carried out in P2 and relating the pressure difference to the strength of the vortex sheet created by the flag, it is possible to derive a single equation for the shape $S(x)$ of a flag in a wind of small angle α to the x-axis. It is shown in [**82**] that $S(x)$ satisfies the (nondimensional) equation

$$(5) \qquad \frac{1}{\pi}⨍_0^1 \frac{S''''(t)}{t-x}\,dt = \kappa(S'(x) - \alpha)$$

where

$$\kappa = \frac{2\rho L^3 U^2}{\gamma}.$$

Here L denotes the length of the flag, ρ and U are respectively the density and speed of the oncoming flow and γ is the flexural rigidity (kg m^2/s^2) of the flag. The flag may be regarded as being either clamped (zero slope) or "hinged" (zero bending moment) at the flag pole to which it is attached. Suitable boundary conditions for (5) are thus $S(0) = 0$, $S''(1) = S'''(1) = 0$, the standard Kutta condition $S''''(1) = 0$, and $S'(0) = 0$ (clamped flag), $S''(0) = 0$ (hinged flag).

Equation (5) is also linear, but it requires numerical techniques that are different from those that may be appropriate to P1 and P2, for not only is it a fourth order equation, but it is posed on a finite range. As far as numerical comparisons are concerned, the equation for the hinged flag has the obvious solution $S(x) = \alpha x$, and though no closed form solutions are known for the clamped case, a number of easily-analyzed asymptotic limits are available for comparison purposes. The numerical solution of (5) was carried out using an *ad hoc* finite difference method in [**29**]. Though the computed solutions appeared to be physically realistic, no error or convergence analysis was given and much more accurate and efficient numerical schemes almost certainly exist.

P4 *Childress slender wake equation.* Steady, two-dimensional solutions of Euler's equations that contain closed regions of constant vorticity were studied by Childress [**13**]. In particular, he considered the flow of an inviscid incompressible stream (with density ρ and undisturbed velocity $U_\infty \hat{\mathbf{e}}_x$, where $\hat{\mathbf{e}}_x$ is a unit vector in the x-direction) down a step formed by the line segments $\{x \leq 0,\ y = h\}$, $\{x = 0,\ 0 \leq y \leq h\}$ and $\{x > 0,\ y = 0\}$, under the assumption that the oncoming flow separated from the top of the step $(0, h)$ and produced a downstream cavity. His asymptotic model, valid for slender eddies, was derived by coupling the lubrication-theory limit of the vorticity equation

$$\nabla^2 \psi = -\,\omega_0$$

(where ψ denotes the stream function and ω_0 the constant vorticity in the cavity) to a potential outer flow via continuity of pressure according to thin aerofoil theory (see, for example [**78, 94**]) across an unknown dividing streamline $y = S(x)$. Thus far, we have presented each problem in a sanitized nondimensional version. In this case, however, it is instructive to first specify the basic model and boundary conditions as this will illustrate the sorts of problem specification difficulty that frequently arise with such equations. In dimensional form, the modeling produces the equation

$$\frac{\rho U_\infty^2}{\pi} \fint_0^L \frac{S'(t)}{t-x}\, dt = b + \frac{1}{2}\rho U_\infty^2 - \frac{\rho \omega_0^2 S^2(x)}{8}. \tag{6}$$

Here b denotes the jump in the Bernoulli constant across the cavity boundary $y = S(x)$, and it has been assumed that the separation streamline reattaches at $(L, 0)$ where $h/L \ll 1$. It is easily shown that the pressure can only remain finite if the flow separates smoothly (i.e. tangentially) from the top of the step and reattaches smoothly at $x = L$: on physical grounds, the correct boundary conditions are therefore

$$S(0) = h, \quad S(L) = 0, \quad S'(0) = 0, \quad S'(L) = 0. \tag{7}$$

It is not initially clear, however, whether these conditions lead to an under- or an over-specified problem. The equation is first order, and so, as we have seen, is likely to require two boundary conditions. Moreover, the parameters L, ω_0 and b are all unknown and must be found as part of the problem. Although it therefore now seems as though there might be too few boundary conditions, an extra relationship between the unknown parameters is concealed in (6). The easiest way to investigate this is to multiply both sides of (6) by $S'(x)$ and integrate from 0 to L. The integral on the left-hand side of (6) is evidently zero, and the right-hand side may now be integrated. Using the boundary conditions (7) now gives

$$b + \frac{1}{2}\rho U_\infty^2 = \frac{\rho \omega_0^2 h^2}{24}.$$

This trick (which here amounts to a global force balance) yields useful results for many SIDEs with a Cauchy kernel. (It is essential though that the necessary double integral exists, and for this reason no progress of this sort can be made on P1 or P2.)

After nondimensionalizing x with L and S with h, the problem becomes

$$\frac{1}{\pi} \fint_0^1 \frac{S'(t)}{t-x}\, dt = \mu(1 - 3S^2(x)), \quad 0 < x < 1 \tag{8}$$

where $\mu = \omega_0^2 Lh/(24\,U_\infty^2)$. The boundary conditions are now

$$S(0) = 1, \quad S(1) = 0, \quad S'(0) = 0, \quad S'(1) = 0$$

and it may be shown that if one of the derivative boundary conditions is satisfied, then the other will automatically hold. Essentially therefore there are three boundary conditions and one unknown parameter μ. The equation involves a single derivative, so the problem specification appears to be in order. P4 furnishes us with our first nonlinear equation; no exact solutions to the problem are known, though some (not very informative) asymptotic limits of (8) may be analyzed. Numerical solutions were successfully computed in [**13**] using an *ad hoc* iterative numerical method.

P5 *Slot-film cooling equation.* In most modern jet engines the gas Turbine Entry Temperature (TET) exceeds the melting temperature of the alloy from which the turbine blades are manufactured. To protect the blades a film of cool air is injected into the flow through slots or holes in the blade surface. An optimization problem naturally arises, for if the injection is too weak the cooling effect will be insufficient, while if the injection is too strong the cool air will penetrate too far into the cross flow and have no effect. Slot film cooling was modeled in [**28**] by coupling a potential flow model for the flow in the injected film with a thin aerofoil theory representation of the cross flow. For weak injection rates, it can be shown that the free streamline $y = S(x)$ that separates the injected coolant flow from the cross flow satisfies the (nondimensional) NLSIDE

$$\frac{1}{\pi} ⨍_0^\infty \frac{S'(t)}{t-x}\,dt = \begin{cases} \dfrac{1}{2} & 0 < x < 1 \\[2ex] \dfrac{1}{2} - \dfrac{M^2}{2S^2(x)} & 1 < x < \infty \end{cases} \tag{9}$$

with boundary conditions $S(0) = S'(0) = 0$, $S(\infty) = M$.

This example is somewhat more intricate than P3, for not only is the range semi-infinite, so that allowance must be made for the behavior of S at infinity, but it may also be shown that

$$S \sim S(1) + O((1-x)\log(1-x))$$

when $x \sim 1$. S thus remains finite, but has infinite slope at an internal point of the domain. We further note that (9) is first order and, arguing as for P1, we therefore expect to specify two boundary conditions. Here though there are three, for in addition to determining $S(x)$, the nondimensional mass flow M of the injected fluid must be found. In actual fact, more information may be obtained about M by using the simple trick that was applied in P4, namely, multiplying both sides of (9) by $S'(x)$ and integrating with respect to x from 0 to ∞. This gives

$$0 = \frac{1}{2}\int_0^1 S'(x)\,dx + \frac{1}{2}\int_1^\infty \left(S'(x) - \frac{M^2 S'(x)}{S^2(x)}\right) dx$$

and thus $M = S(\infty) = 2S(1)$ and, if required, M may be removed completely from the problem.

As far as the asymptotic behavior of the solution is concerned, we note that (9) may be inverted to yield

$$S'(x) = \frac{\sqrt{x}}{2\pi} \int_1^\infty \frac{M^2}{S^2(t)\sqrt{t}(t-x)}\,dt,$$

or, if we write $S = M^{2/3}T$,

$$T'(x) = \frac{\sqrt{x}}{2\pi} \int_1^\infty \frac{1}{T^2(t)\sqrt{t}(t-x)}\,dt.$$

If T is monotonic and bounded above, then it is relatively simple to obtain the asymptotic estimates for the solution in the form

(10)

$$T(x) \sim x^{3/2} \quad \text{as } x \to 0, \quad \text{and} \quad T(x) \sim T(\infty) - \frac{T^4(\infty)}{\pi x} \quad \text{as } x \to \infty.$$

This problem was solved in [**28**] using direct iteration; the relatively slow approach of the solution to $T(\infty)$ implied by (10) has important numerical implications as it demands that accurate numerical estimates must be made for large x.

P6 *Fluid suction equation.* Polluted fluid lying below a clean stream flows steadily along the x-axis: a porous suction slot occupying the region $0 \le x \le L$ is to be used to remove the unwanted contaminant.

For $x > L$ the flow is constrained by a wall of a given height. The suction is arranged so that a desired amount of pollutant can be removed. Using similar reasoning to that employed to derive the equations for P4 and P5, it may be shown that the nondimensional height $S(x)$ of the final polluted streamline satisfies the equation

$$\frac{1}{\pi} \fint_0^1 \frac{S'(t)}{t-x}\, dt = \frac{\theta}{S(x)} - 1, \quad 0 < \theta \leq 1, \tag{11}$$

with boundary conditions $S(0) = S_0,\ S(1) = S_1$. The total amount of fluid that is removed from the oncoming stream is determined by the suction strength $\theta > 0$ and the height S_1 of the downstream wall. As we shall demonstrate in Section 4, (11) is a particularly interesting problem since under certain conditions the solution may not exist.

P7 *Bissett/Spence lubrication equation.* The line contact problem of elastohydrodynamic lubrication theory was studied in [**6**] in an asymptotic limit corresponding to slow bearing rotation speeds. The general structure of the flow under the bearing is complicated, for it transpires that a transition layer where rapid changes take place separates distinct inlet and contact zones, before the flow leaves the bearing in a downstream exit layer. In both the transition and exit zones lubrication theory may be coupled to standard contact problem theory for plane elasticity (see, for example [**77**]). The result is that the flow in each region is determined by an NLSIDE. Here, we specifically consider the transition region, within which the nondimensional film thickness $S(x)$ satisfies

$$\frac{\sigma}{\pi} \fint_{-\infty}^{\infty} \frac{S(t)}{(1+S(t))^3} \frac{dt}{t-x} = S''(x). \tag{12}$$

In (12) σ is a nondimensional parameter related to the scaled exit film thickness, which may be regarded for the purposes of this discussion as known (though in fact it must be determined from another, separate problem). The equation is second order and the highest derivative does not appear in the singular integral operator; we therefore expect to have to prescribe two boundary conditions. As is often the case when the range is infinite, the boundary conditions amount to matching conditions and assert that

$$S \sim x^{-1/2} \quad x \to +\infty, \qquad S \sim \frac{4}{3}\sigma(-x)^{3/2} \quad x \to -\infty.$$

This problem presents some formidable challenges. It is nonlinear and second order, the boundary conditions are somewhat awkward and the infinite range means that proper account must be taken of contributions from both $-\infty$ and ∞.

As with nearly all of the equations reviewed in this collection, there are alternative ways to present the problem that may be more or less convenient from a numerical point of view. In [**6**] (12) was inverted to yield

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{S''(t)}{t-x}\, dt = -\frac{\sigma S(x)}{(1+S(x))^3}$$

subject to the boundary conditions $S(0) = 1/2$, $S''(-\infty) \sim \sigma(-x)^{-1/2}$ (and implicitly $S'(0) = 0$). A numerical solution was then sought by transforming the problem onto the finite region $[0, \pi]$ and expanding the solution in a series of suitably chosen trigonometrical functions for which the required integrals are known explicitly (see also [**87, 88**]). The resulting nonlinear equations for the coefficients in the Fourier expansion were then solved using the Powell hybrid method (see, for example [**83**]). Needless to say, the choice of coordinate transformation, the form of the expansion and the numerical methodology for truncating the Fourier series all have a profound effect on the accuracy of the final numerical solution.

P8 LMFBR *boiler tube equation.* In a Liquid Metal Fast Breeder Reactor (LMFBR) water is heated in pipes by surrounding the pipe with a countercurrent liquid metal heat source. The water enters the pipe at ambient temperature and travels up the pipe. As it is heated, bubbles develop and coalesce into "slug flow" as the water boils. After transition through a complicated churn-turbulent region of two-phase gas/liquid flow, an "annular" two-phase flow is established. This is characterized by a fast-moving column of water vapor in the center of the pipe which is surrounded by a thin layer of superheated water on the pipe wall. This annular flow region takes up most of the length of each heat exchanger pipe. Eventually, the heat transfer from the liquid metal through the pipe wall triumphs and the last vestiges of liquid are turned to steam, which then drives turbines to produce electricity. From a practical point of view, it is important to be able to predict the position of the "dryout" point where the liquid layer vanishes as thermal stresses at this point may degrade the integrity of the pipe wall. This problem was considered in [**76**], where a thin layer analysis in the fluid region was coupled to a high-speed potential flow model in the gas core. The thermal problem was solved in the fluid region, where the temperature gradient drives a Stefan problem that determines how quickly the superheated fluid loses heat to allow it to boil away as steam and take its place in the gas core flow. When the flow is steady ([**76**] also treated unsteady cases) it was shown that the nondimensional location $y = S(x)$ of the boundary separating the fluid region from the gas satisfies the singular integro-differential equation

$$\frac{\theta}{\pi}\left(\int_0^1 \frac{S'(t)}{t-x}\,dt\right)_x = \frac{Q}{S^3(x)} + \frac{\tau}{2S(x)}, \tag{13}$$

where Q, τ and θ are all nondimensional (and, in the most general case all $O(1)$) parameters; Q characterizes the wall heating and determines how much evaporation takes place, τ measures the importance of the shear stress exerted by the gas core on the fluid region and θ measures the scaled order of magnitude of the pressure in the gas flow relative to that in the fluid flow. The boundary conditions are

$$S'(0) = 0, \quad S(1) = 0, \quad S(0) = A$$

where A is a known constant related to the pressure in the pipe at the onset of annular flow. Although (13) is somewhat similar to the water drop equation described in P9, we shall see in Section 6 that it possess some properties that may lead to serious numerical complications. It also contains an extra challenge, namely that the singular integral is itself differentiated.

P9 *Air–blown water drop equation.* The ability of an upward flow of air to support a thin liquid layer against gravity on a plane (as, for example on a car windscreen in a rain shower) was examined in [**62**]. In this study, the standard lubrication theory equations were solved to determine the flow inside a long, thin drop with free surface $h(x)$ and length L. The assumption that $h'(x) = O(\varepsilon)$ provides the small parameter $\varepsilon \ll 1$ in the problem. By coupling the pressure inside the drop to thin-aerofoil theory and including the effects of gravity and surface tension, $h(x)$ may be shown to satisfy the nondimensional NLSIDE

$$\lambda\left(\int_0^1 \frac{S'(t)}{t-x}\,dt\right)_x - \beta S'''(x) + S'(x) + \gamma = \frac{1}{S(x)}. \tag{14}$$

Here x has been scaled with L and $h(x) = h_0 S(x)$ where $h_0 = \sqrt{3\tau L/2\rho_w g}$. The other nondimensional parameters in the problem are $\lambda = \rho U_\infty^2/(\pi L \rho_w g)$, $\beta = \sigma/(\rho_w g L^2)$ and $\gamma = L\alpha/h_0$. The parameter τ is the shear stress produced by the flow over the drop, σ denotes the surface tension (N/m), ρ_w and ρ the densities of the water in the drop and the free stream (of speed U_∞) respectively, g is the acceleration due to gravity and α is the angle (presumed small) of the plane to the horizontal. As far as boundary conditions are concerned, it was assumed in [**62**] that $\beta \ll 1$ so that the third derivative term was absent from (14). As usual, it is now necessary to consider carefully which of the parameters are known and which are unknown, so that suitable boundary conditions may be specified. Probably the easiest way of understanding the details of the problem specification is to note that, so long as the physical properties of the air and the water and the shear stress τ are known, then once λ has been determined, L is then given by the relationship

$$L = \frac{\rho U_\infty^2}{\pi \lambda \rho_w g}$$

and the problem is essentially solved. We expect that (14) (with $\beta = 0$) will require three boundary conditions, and it transpires that these are

$$S(0) = 0, \quad S(1) = 0, \quad S'(1) = -\lambda^{-1/2}. \tag{15}$$

The drop therefore attaches linearly at its downstream end $x = 1$ but has infinite slope at its upstream end $x = 0$, where $S(x) \sim \sqrt{2x}$. Though thin aerofoil theory predicts that the behavior at $x = 0$ will lead to an infinite pressure, it is easy to see that the inclusion of the hitherto neglected surface tension term in (14) gives rise to a boundary layer of width $O(\beta^{1/2})$ that allows attachment with finite slope at the nose of the drop.

The problem is a challenging one numerically, since not only is the problem nonlinear, but for arbitrary values of the physical parameters of the problem, λ plays the role of an eigenvalue in that it must take a particular value if a solution is to exist to (14) subject to (15). Any successful numerical scheme must therefore not only be capable of determining λ, but also be able to cope with the infinite slope at the nose of the drop. In [**62**] the problem was first regularized by setting

$y = \sqrt{x}$ and $S(x) = \sqrt{2x}H(y)$ and then further simplified by defining $G(y) = H(y) + yH'(y)$. The resulting NSLIDE was then solved by using discretization and collocation to produce a system of nonlinear algebraic equations which were solved using the Powell hybrid method (see [**83**]). The numerical results showed that two values of λ were possible for $\alpha < 0.46\lambda^{-1/2}\varepsilon$. It was not clear which, if either, of these values was stable. This result also suggested that wind-supported drops are not possible for sufficiently large α.

This problem is particularly illuminating as issues regarding problem specification, boundary conditions, asymptotic behavior of the solution, boundary layers, regularization, nonuniqueness and nonexistence that are so commonly encountered in the analysis of NLSIDEs are all present.

P10 *Cavitating Prandtl-Batchelor aerofoil.* A model of an aerofoil with a recirculating Prandtl-Batchelor region behind a spoiler was posed in [**99**] as a paradigm for the flow above a stalled aerofoil. In this problem, because the obstruction to the flow not only has finite thickness but also produces nonzero lift, there are two unknown functions, namely the height of the cavity $y = S(x)$ above the aerofoil surface, and the cavity vorticity $v(x)$. Coupling the cavity flow with a thin aerofoil theory model of the oncoming stream by insisting in the normal way that the pressure is continuous across the cavity boundary yields the two NLSIDEs

$$\frac{1}{2\pi}\int_0^\beta \frac{S'(t)}{t-x}\,dt = 2\pi v(x) + \lambda - \frac{(\Gamma S(x))^2}{8}, \tag{16}$$

$$\int_0^1 \frac{v(t)}{t-x}\,dt = \frac{1}{2}\left(h'(x) + \frac{S'(x)}{2}\right). \tag{17}$$

Here the (known) function $h(x)$ is determined by the aerofoil geometry, β is the (unknown) x-ordinate at which the cavity reattaches to the aerofoil, λ is the perturbation Bernoulli constant in the cavity and Γ is the nondimensional constant cavity vorticity. Once again, it is not immediately clear what is known and what is unknown. It transpires, however, that if the reattachment point β is specified, then the quantities λ and Γ are uniquely determined. We therefore require one boundary condition for v and four for S. This makes excellent physical sense, as the Kutta condition at the trailing edge of the aerofoil requires that $v(1) = 0$, the separation points give $S(0) = h_0$ and $S(\beta) = 0$, and the requirement that the pressure is finite at separation and reattachment gives $S'(0) = h_1$ and $S'(\beta) = 0$ (where the known constants h_0 and h_1 are related to the aerofoil geometry).

The equations (16) and (17) pose a challenging problem; they are a nonlinear system and include unknown parameters that cannot be eliminated. A numerical solution was calculated in [**99**] using an *ad hoc* iterative method. It seems however that nothing at all has appeared in the literature concerning the theory of numerical schemes for systems of NLSIDEs such as (16) and (17).

4. The crucial importance of well-posedeness.

4.1 *Regularization.* This section is devoted to further analysis of P6 and equation (11), the purpose of the analysis being to illustrate how the well-posedness or otherwise of NLSDIEs may depend delicately on the boundary conditions. To aid this analysis, let us first regularize (11) by inverting the finite range Hilbert transform. This may be done by using standard techniques (see for instance Muskhelishvili [**77**]). We obtain

$$S'(x) = -\frac{1}{\pi} \int_0^1 \sqrt{\frac{t(1-t)}{x(1-x)}} \left(\frac{\theta}{S(t)} - 1 \right) \frac{dt}{t-x} + \frac{C}{\sqrt{x(1-x)}} \tag{18}$$

where C is an arbitrary constant that must be chosen to satisfy the boundary conditions. Integrating (18) with respect to x from 0 to x gives

$$S(x) - S(0) = -\frac{1}{\pi} \int_0^1 \sqrt{t(1-t)} \left(\frac{\theta}{S(t)} - 1 \right) \times \left[\int_0^x \frac{1}{\sqrt{s(1-s)}\,(t-s)}\, ds \right] dt + C \int_0^x \frac{1}{\sqrt{s(1-s)}}\, ds$$

so that

$$S(x) - S(0) = -\frac{1}{\pi} \int_0^1 \left(\frac{\theta}{S(t)} - 1 \right) \log \left| \frac{\sqrt{x(1-t)} + \sqrt{t(1-x)}}{\sqrt{x(1-t)} - \sqrt{t(1-x)}} \right| dt + \frac{C\pi}{2} + C \sin^{-1}(2x-1).$$

Since $S(0) = S_0$ and $S(1) = S_1$, we find that $S_1 - S_0 = C\pi$, and hence (11) may be rewritten in the Fredholm integral equation form

$$S(x) + \frac{\theta}{\pi} \int_0^1 \frac{k(x,t)}{S(t)}\, dt = g(x) \tag{19}$$

where

$$k(x,t) = \log \left| \frac{\sqrt{x(1-t)} + \sqrt{t(1-x)}}{\sqrt{x(1-t)} - \sqrt{t(1-x)}} \right|,$$
$$g(x) = \sqrt{x(1-x)} - \frac{2}{\pi}\,(S_1 - S_0) \sin^{-1} \sqrt{1-x} + S_1$$

and use has been made of the identity $\sin^{-1}(2x-1) = (\pi/2) - 2\sin^{-1}\sqrt{1-x}$.

4.2 *Non-existence of a solution.* If $S_0 > 0$ and $S_1 = 0$, then problem (11) has no Hölder continuous solution, as we shall prove below. We start by noting that

Lemma 4.1. *Let $S(x)$ be a solution of* (11) *with S a Hölder continuous function on $[0,1]$, and assume that $S_0 > 0$. Then*

1. $S(x) > 0$ for all $x \in (0,1)$;
2. $S(x) \le g(x)$ for all $x \in [0,1]$.

Proof. 1. Note first that, as S is a solution of (11), the left-hand side of (11) is finite for $x \in (0,1)$. This implies that $S(x)$ cannot vanish for $x \in (0,1)$. Since $S(0) > 0$, it must therefore be true that $S(x) > 0$ for $x \in (0,1)$.

2. Since S is Hölder continuous, the regularization process resulting in (19) can be carried out (see [**77**, p. 249]). Hence, if S is a solution of (11) it is also a solution of (19). The function $k(x,t)$ is evidently always strictly positive for $x \in (0,1)$ and $t \in (0,1)$, and since $S(x) > 0$ for all $x \in (0,1)$, we see from (19) that we must have $S(x) \le g(x)$ for $x \in (0,1)$. Since $S(0) = S_0 = g(0)$ and $S(1) = S_1 = g(1)$, the second part of Lemma 4.1 is proved.

We are now in the position to prove our main result.

Theorem 4.1. *When* $0 < \theta \leq 1$ *and* $S_1 = 0$*, problem* (11) *and its regularization* (19) *have no Hölder continuous solution.*

Proof. Assume that $x \in [0,1]$ and set $S_1 = 0$. For $S_0 > 0$ and $\theta = 0$, we see from (19), $S(x) = g(x)$. Since $\sin^{-1}\sqrt{1-x} \leq (\pi/2)\sqrt{1-x}$ for all $x \in [0,1]$,

$$g(x) \leq A\sqrt{1-x}, \quad A = 1 + S_0.$$

Then since $k(x,t) \geq 0$ for all $(x,t) \in [0,1] \times [0,1]$, for a solution satisfying (19) we must have

$$\int_0^1 k(x,t)\,\frac{dt}{S(t)} \geq \int_0^1 k(x,t)\,\frac{dt}{g(t)} \geq \frac{I(x)}{A} \tag{20}$$

where

$$I(x) = \int_0^1 \frac{k(x,t)}{\sqrt{1-t}}\,dt.$$

Integrating $I(x)$ by parts yields

$$\begin{aligned} I(x) &= 2\sqrt{x(1-x)} \fint_0^1 \frac{dt}{\sqrt{t}\,(x-t)} \\ &= 2\sqrt{1-x}\log\left|\frac{1+\sqrt{x}}{1-\sqrt{x}}\right| \\ &\geq 2\sqrt{1-x}\log\left|\frac{1}{1-x}\right|. \end{aligned} \tag{21}$$

From (19), (20) and (21) we now have

$$g(x) - S(x) \geq \frac{2\theta}{\pi A^2}\,\log\left|\frac{1}{1-x}\right| g(x).$$

For every $\theta > 0$ and $S_0 > 0$, there exists an x_θ such that

$$\frac{2\theta}{\pi A^2}\,\log\left|\frac{1}{1-x}\right| > 1, \quad \text{for} \quad x_\theta < x < 1.$$

In this range $g - S \geq g$, i.e., $S \leq 0$ which contradicts Lemma 4.1 and proves the theorem.

Consider further the case when $S_1 \neq 0$. Now we have

$$g(x) = \sqrt{x(1-x)} - \frac{2}{\pi}(S_1 - S_0)\sin^{-1}\sqrt{1-x} + S_1$$

and significantly $g(x)$ does not now tend to zero as $x \to 1$. Since the nonexistence proof of Theorem 4.1 above seems to depend crucially on bounding the function $g(x)$ by $A\sqrt{1-x}$, it seems most unlikely that the proof could be adapted to include cases where $S_1 \neq 0$. Indeed, the work in subsection 5.2 below strongly suggests that for $S_1 > 0$ a solution exists and may be computed with little trouble.

5. Numerical solution.

5.1 *Asymptotic analysis.* We now seek an asymptotic form of the solution of (11) to assist with the design of our numerical schemes. From (19), we know that when $\theta = 0$ the function $g(x)$ is a solution, and therefore

$$S(x) = \sqrt{x(1-x)} - \frac{2}{\pi}(S_1 - S_0)\sin^{-1}\sqrt{1-x} + S_1, \quad \theta = 0.$$

In the special case $\theta = 0$, therefore, we have

$$S \sim S_0 + \sqrt{x}\left(1 + \frac{2(S_1 - S_0)}{\pi}\right) + O(x^{3/2}), \quad x \to 0$$
$$S \sim S_1 + \sqrt{1-x}\left(1 - \frac{2(S_1 - S_0)}{\pi}\right) + O((1-x)^{3/2}), \quad x \to 1,$$

so that it is clear that $S'(x)$ is unbounded at both ends of the range.

Is this asymptotic behavior retained when $\theta > 0$? To answer this, we consider an arbitrary $\theta > 0$ and seek an asymptotic solution as $x \to 0$ of the form

$$S(x) = S_0 + K_1(\theta)x^m, \quad x \to 0.$$

Choosing R so that $0 \leq x \ll R \ll 1$ and using the definition of the Cauchy principal value, we see that the integral term in (11) is given by

$$mK_1(\theta)\lim_{\xi\to 0}\left\{\int_0^{x-\xi}\frac{s^{m-1}}{s-x}\,ds+\int_{x+\xi}^{R}\frac{s^{m-1}}{s-x}\,ds\right\}+\int_R^1\frac{S'(s)}{s-x}\,ds$$
$$=mK_1(\theta)\lim_{\xi\to 0}\left\{-\int_0^{x-\xi}\frac{s^{m-1}}{x}\,(1+s/x+s^2/x^2+\cdots)\,ds\right.$$
$$\left.+\int_{x+\xi}^{R}\frac{s^{m-1}}{s}\,(1+x/s+x^2/s^2+\cdots)\,ds\right\}+\int_R^1\frac{S'(s)}{s-x}\,ds,$$

and on integration it is now straightforward to see that as $\xi\to 0$ the only instance when terms $O(x^{-p})$, $p>0$, cancel (so that the integral is finite) is when $m=1/2$. It is now easy to show that, in order to balance the terms at next order as $x\to 0$, we must include a logarithmic term, and thus

$$S(x)\sim S_0+K_1(\theta)x^{1/2}+K_2(\theta)x^{3/2}\log x+O(x^{3/2}),\quad x\to 0$$

where $K_1(\theta)$ and $K_2(\theta)$ may be expressed in terms of quadratures involving $S(x)$. Assuming the validity of the differentiation, this shows that $S'(x)$ behaves like $x^{-1/2}$ as $x\to 0$, a fact that will be used to design our numerical schemes. (Near to $x=1$, it is possible to show by a change of variables that the same behavior applies.)

5.2 *Numerical procedures.* In this section we describe two numerical procedures to solve problem (11). One is based on the numerical approximation of equation (11) itself and the other is based on approximating the regularized equation (19).

5.2.1 *Global collocation.* We begin by solving (11) directly. In view of the asymptotic results of subsection 5.1, a solution is sought in the form

$$F'(w)=(1-w^2)^{-1/2}\sum_{j=0}^{\infty}a_jT_j(w)\tag{22}$$

where

$$F(w)=S\left(\frac{w+1}{2}\right),\quad w=2x-1\in[-1,1],\quad \text{for}\quad x\in[0,1],$$

and $T_j(w)$ are the Tchebyshev polynomials of the first kind defined by $T_j(w) = \cos(j\cos^{-1} w)$. With the same change of variables the SIDE (11) becomes

$$\frac{\theta}{F(w)} - 1 = \frac{2}{\pi} \fint_{-1}^{1} \frac{F'(u)}{u-w}\, du. \tag{23}$$

Integrating (22) and imposing the boundary conditions $F(-1) = S_0$ and $F(1) = S_1$ so that $a_0 = (S_1 - S_0)/\pi$, we obtain

$$F(w) = S_1 + \frac{S_0 - S_1}{\pi} \cos^{-1} w - \sum_{j=1}^{\infty} \frac{a_j}{j} \sin(j\cos^{-1} w). \tag{24}$$

Substituting (22) and (24) into (23) now yields

$$\theta \left\{ S_1 + \left(\frac{S_0 - S_1}{\pi}\right) \cos^{-1} w - \sum_{j=1}^{\infty} \frac{a_j}{j} \sin(j\cos^{-1} w) \right\}^{-1} - 1 = \frac{2}{\pi} \fint_{-1}^{1} \frac{1}{(1-u^2)^{1/2}(u-w)} \sum_{j=0}^{\infty} a_j T_j(u)\, du. \tag{25}$$

The Tchebyshev polynomials of the first kind T_j and their companions of the second kind U_j satisfy the relation

$$\fint_{-1}^{1} \frac{T_j(u)}{(1-u^2)^{1/2}(u-w)}\, du = \begin{cases} 0 & j = 0 \\ \pi U_{j-1}(w) & j \geq 1, \end{cases} \tag{26}$$

so from (25) and (26) we obtain the final equation for the coefficients a_j,

$$\theta \left\{ S_1 + \left(\frac{S_0 - S_1}{\pi}\right) \cos^{-1} w - \sum_{j=1}^{\infty} \frac{a_j}{j} \sin(j\cos^{-1} w) \right\}^{-1} - 1 = 2 \sum_{j=1}^{\infty} a_j U_{j-1}(w). \tag{27}$$

To compute an approximation to the function $F(w)$, it is necessary to truncate the series (24) after N terms. The N unknowns a_j,

$j = 1, 2, \ldots, N$, are then determined by collocation at N points, i.e., equation (27) is forced to be exactly satisfied at N points chosen from $[-1, 1]$. The N points chosen were the zeros of the Nth Tchebyshev polynomial of the first kind since these points, at least for the linear equation, are known to give a faster rate of convergence (see, for example, [**14, 21**]). The resulting nonlinear system was then solved using the MATLAB routine *fminsearch*.

5.2.2 *Piecewise collocation.* We turn now to the problem of solving the regularized equation (19) by dividing the interval $[0, 1]$ into N equally-spaced intervals bounded by $0 = x_0, x_1, x_2, \ldots, x_N = 1$, and writing the integral equation at the internal points of the intervals as

$$S(x_i) = g(x_i) - I(x_i), \quad i = 1, 2, \ldots, N-1$$

where

$$I(x) = \frac{\theta}{\pi} \int_0^1 \frac{k(x,t)}{S(t)}\, dt.$$

The general plan is now to successively calculate

$$\begin{aligned} \widetilde{S}^{(k)}(x_i) &= g(x_i) - I(x_i) \\ S^{(k+1)}(x_i) &= S^{(k)}(x_i) + \phi(\widetilde{S}^{(k)}(x_i) - S^{(k)}(x_i)) \end{aligned} \tag{28}$$

for $1 \le i \le N-1$ where ϕ is an under-relaxation parameter, chosen before the iterative process starts, and $k = 0, 1, 2, \ldots$, an initial guessed profile for S.

To approximate $I(x_i)$, we assume that S is piecewise constant on each of the intervals $[x_j, x_{j+1})$ and write

$$I(x_i) = \frac{\theta}{\pi} \int_0^1 \frac{k(x_i,t)}{S(t)}\, dt = \frac{\theta}{\pi} \sum_{j=0}^{N-1} \frac{1}{2}\,(1/S_j + 1/S_{j+1}) h(x_i, x_j) \tag{29}$$

where $S_j = S(x_j)$ and

$$\begin{aligned} h(x_i, x_j) &= \int_{x_j}^{x_{j+1}} k(x_i, t)\, dt \\ &= (x_{j+1} - x_i) \log \left| \frac{\sqrt{x_i(1 - x_{j+1})} + \sqrt{x_{j+1}(1 - x_i)}}{\sqrt{x_i(1 - x_{j+1})} - \sqrt{x_{j+1}(1 - x_i)}} \right| \\ &\quad - (x_j - x_i) \log \left| \frac{\sqrt{x_i(1 - x_j)} + \sqrt{x_j(1 - x_i)}}{\sqrt{x_i(1 - x_j)} - \sqrt{x_j(1 - x_i)}} \right| \\ &\quad + \sqrt{x_i(1 - x_i)} \left[\sin^{-1}(2x_{j+1} - 1) - \sin^{-1}(2x_j - 1)\right]. \end{aligned}$$

5.2.3 *Numerical results.* Computations obtained from the piecewise method (28) are shown for 10, 20 and 100 points superimposed in Figure 1. The parameters used were $\theta = 1/2$, $S(0) = 1$ and $S(1) = 1/2$. No numerical relaxation was necessary ($\phi = 1$) and solutions were deemed to have converged when

$$\sum_{i=1}^{N-1} \left(\widetilde{S}_i^{(k)} - S_i^{(k)}\right)^2 < \text{TOL}$$

where the tolerance TOL was chosen to be 10^{-8}. Convergence occurred very quickly, taking only 9 iterations for 10 and 20 points and 10 iterations for 100 points. By examining predicted values for $S(1/2)$, it was clear that the scheme was converging as the number of mesh points increased. The numerical results of Figure 1 also confirm the asymptotic analysis, the square-root behavior at each end of the range being evident.

When the global collocation method (27) was used to solve (11) with $S(0) = 1$, $S(1) = 1/2$ and $\theta = 1/2$, the results were indistinguishable from the converged solution calculated using the piecewise collocation method. This may be regarded as an "easy" case for both of the schemes. When values of $S(1)$ that are close to zero are used, however, things become altogether more awkward. The piecewise collocation method requires a greater number of iterations and more under-relaxation to converge (with $S(0) = 1$ and $S(1) = 0.05$, for example,

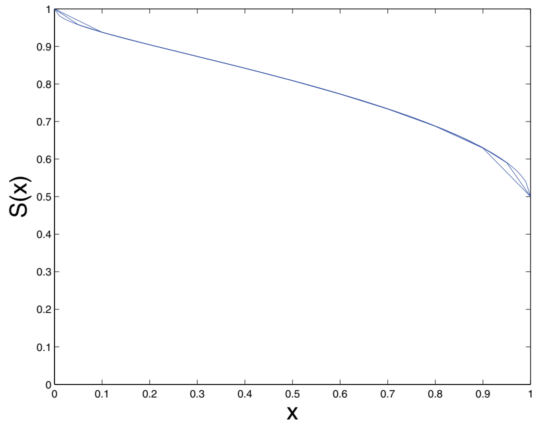

FIGURE 1. Piecewise collocation solution of (19) for $N = 10$, 20 and 100 with $S(0) = 1$, $S(1) = 1/2$ and $\theta = 1/2$.

even computations with only 10 mesh points required $\phi = 0.001$ and took 21562 iterations to converge). For values of $S(1)$ less than about 0.01, this method fails entirely. In contrast to this, the global collocation method (27) continues to produce plausible-looking solutions even when $S(1) = 0$, though we know that for this case no solution exists. As $S(1) \to 0$ both the error and the number of iterations required increase, but without careful monitoring of these parameters one could easily be mislead into suspecting that there is nothing amiss with the problem. This provides further confirmation of our earlier statement that, for the type of equations considered in this study, understanding the existence and uniqueness, asymptotic structure and physical aspects of the solutions are all crucial.

6. An example strategy for obtaining solutions.

6.1 *Preamble.* The previous two sections have highlighted the difficulties in attempting to solve the apparently innocuous NLSIDE (11). Here we were fortunate that we were able to establish some results concerning the nonexistence of the solution under certain circumstances. In general this will not be the case. Nevertheless, it is possible to suggest a tentative strategy for determining credible solutions and techniques for checking that credibility. We shall illustrate our approach using the equation

$$\frac{\theta}{\pi}\left(\fint_0^1 \frac{S'(t)}{t-x}\,dt\right)_x = \frac{(x-1)\dot{m}}{S^3(x)} + \frac{\tau}{2S(x)}, \tag{30}$$

a version of the boiler tube equation from problem P8 where it has been assumed that the liquid metal produces a specific form of heating, namely $Q = (x-1)\dot{m}$ where $\dot{m}$ is a constant.

6.2 *A linear equation.* Before considering the full nonlinear equation, let us examine the simpler linear equation

$$\frac{1}{\pi}\left(\fint_0^1 \frac{S'(t)}{t-x}\,dt\right)_x = 1, \tag{31}$$

subject to the boundary conditions

$$S'(0) = 0, \quad S(1) = 0, \quad S(0) = 1 \tag{32}$$

corresponding to taking $A = 1$ in P8. During our discussion of P1 a linear equation very similar to (31) was solved in closed form; we now wish to consider possible numerical approaches to the solution of the paradigm problem (31), (32).

Suppose that we first proceed using an *ad hoc* method and discretize the interval $[0,1]$ by dividing it into n equally spaced subintervals $[\xi_j, \xi_{j+1}]$ where $\xi_j = j/n$, $0 \le j \le n-1$, are the mesh points and n denotes the number of mesh points. We now decompose the integral in (31) into the sum of n integrals, apply the trapezoidal rule to each of those integrals, perform a central difference for the derivative and collocate, giving a system of linear equations. The solution of these equations would be a routine matter, and this method has the benefit of simplicity and speed. Unfortunately (as shown in [**76**]), these advantages are outweighed by the fact that this method can easily be shown to produce solutions that do not converge as the grid is refined. This may be easily confirmed, for (31) admits the analytic solution (see, for example [**77, 80, 81**])

$$S(x) = \left(\frac{2}{\pi} - \frac{x}{2}\right)\sqrt{x(1-x)} - \frac{1}{\pi}\sin^{-1}(2x-1) + \frac{1}{2}. \tag{33}$$

This experience suggests that we should examine the problem more closely before designing a numerical scheme. We first note that (33) implies that, near $x = 1$, $S(x) \to 0$ like $\sqrt{1-x}$. The slope of $S(x)$ thus becomes unbounded as $x \to 1$, and it is this that disrupts the simple numerical scheme discussed above.

Obviously the problem requires some sort of regularization, so in this case we set

$$S(x) = T(y), \quad \text{where} \quad y^2 = 1 - x. \tag{34}$$

Equation (31) now becomes

$$\frac{1}{\pi}\left(⨍_0^1 \frac{T'(u)}{y^2 - u^2}\, du\right)_y = 2y, \tag{35}$$

subject to the boundary conditions

$$T(0) = 0, \quad T'(1) = 0, \quad T(1) = 1. \tag{36}$$

As is shown in [**76**], the conventional numerical discretization discussed above now displays convergence as the grid is refined. Of course, a price has been paid for this convergence, for not only was the regularization necessary, but the amount of algebra required to obtain the relevant set of linear equations is now greatly increased.

6.3 *Asymptotics and regularization.* We now return to the full problem (30). At the upstream end $x = 1$ (the dryout point, see P8) the fluid layer thickness $S(x)$ must vanish. Therefore, the nonlinear term $\dot{m}/S^3(x)$ which dominates the righthand side of (30) must be balanced by at least one other large term in the equation. We shall assume that the shear stress is negligible and that the required contribution comes from the Cauchy integral in a small region near $x = 1$. Thus we assume that $S(x) \sim A(1-x)^p$ for some positive constants A and p ($p < 1$). An asymptotic balance ultimately gives

$$S(x) \sim \left(\frac{4\pi\dot{m}}{\theta}\right)^{1/4} (1-x)^{1/2}, \quad \text{as} \quad x \to 1 \tag{37}$$

(correct to powers of $1 - x$). Encouraged by the linear problem of subsection 6.2, we write

$$S(x) = \gamma T(y), \quad y^2 = 1 - x, \quad (\xi^2 = 1 - t)$$

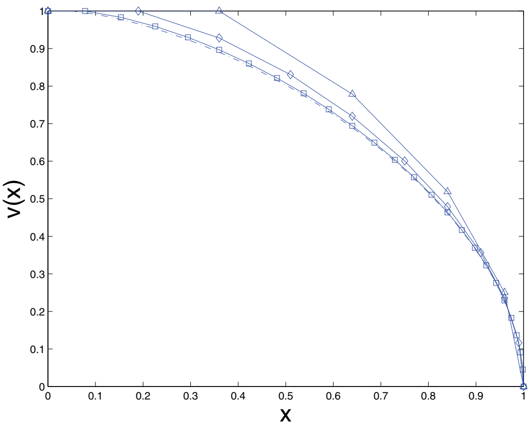

FIGURE 2. Graph of $v(x)$ for $n = 5$ (triangles), 10 (diamonds), 25 (squares), 40 (broken line) and $\dot{m} = 1$, $\theta = 1$ and $\tau = 2$.

where $\gamma = (4\pi\dot{m}/\theta)^{1/4}$ to produce the regularized version

(38)

$$\left(\int_0^1 \frac{T'(\xi)}{y^2-\xi^2}\,d\xi\right)_y = \left(\frac{\pi y^2}{\dot{m}\theta}\right)^{1/2}\left[-\frac{y^2}{2T^3(y)}\left(\frac{\dot{m}\theta}{\pi}\right)^{1/2} + \frac{\tau}{2T(y)}\right],$$

subject to the boundary conditions

$$T(0) = 0, \quad T'(1) = 0, \quad T(1) = \left(\frac{\theta}{4\pi\dot{m}}\right)^{1/4}.$$

6.4 *Numerical method and results.* Following [**76**] a mesh $\xi_j = j/n$ is defined on the interval $[0,1]$ with $T'(\xi)$ assumed constant in each subinterval $[\xi_j, \xi_{j+1}]$, that is, the function $T(\xi)$ is approximated by linear functions in each subinterval. The pressure gradient

$$p_y = \left(\int_0^1 \frac{T'(\xi)}{y^2-\xi^2}\,d\xi\right)_y$$

is evaluated by finite differences. Collocating (38) at the mesh points results in the $(n-2)\times(n-2)$ set of nonlinear algebraic equations given by

$$\sum_{j=0}^{n-1}(T_{j+1}-T_j)\left[\frac{1}{2i+1}\log\left|\frac{(2i+2j+3)(2i-2j+1)}{(2i-2j-1)(2i+2j+1)}\right|\right.$$
$$\left.-\frac{1}{2i-1}\log\left|\frac{(2i+2j+1)(2i-2j-1)}{(2i-2j-3)(2i+2j-1)}\right|\right]$$
$$=\frac{i}{n^4}\left(\frac{\pi}{\dot{m}\theta}\right)^{1/2}\left[-\frac{i^2}{2n^2T_i^3}\left(\frac{\dot{m}\theta}{\pi}\right)^{1/2}+\frac{\tau}{2T_i}\right],\quad 1\le i\le n-2$$

and $T_0 = 0$, $T_n = T_{n-1} = (\theta/4\pi\dot{m})^{1/4}$. The values of $v_j = (4\dot{m}\pi/\theta)^{1/4}T_j$ were then plotted against $x_j = 1-y_j^2$, $0\le j\le n$ as displayed in Figure 2.

Of course, one cannot be certain that one has the correct solution, or even if a solution exists at all. There are however a number of checks: the residuals may be computed especially with decreasing mesh spacing; the solution may be computed on different grids and the approximate convergence rate computed. Figure 2 seems to confirm that a solution exists and that the numerical method converges to this solution as the mesh is refined.

7. Discussion and conclusion. The paper has attempted to provide a reasonably comprehensive review of both linear and non-linear Cauchy integral and integro-differential equations as they have appeared in the literature. Most have been little studied and some have defied even a numerical solution. Almost no convergence arguments have ever, to the authors' knowledge, been provided for any numerical method that solves nonlinear Cauchy integral equations (but see [**59**]). Thus, there is scope for both analysts and numerical analysts. The challenges include the well-posedeness of the problem, the derivation of asymptotic results, the design of a (universal) numerical algorithm and its associated convergence proof and, ultimately, the translation of the algorithm into public-domain or commercial software. The 20 equations detailed above and in the Appendix have been written down with their boundary conditions and associated parameters and their ranges. We see them, or at least a subset of them, as also providing a test set against which new algorithms may be gauged. However, a word of caution is required, for we cannot guarantee the correctness of all of these equations either from a modeling or a well-posedness point of view. Evidently the latter will not in general be possible until analysts develop a general theory of well-posedness, or, if that proves impossible, are able to demonstrate existence and uniqueness on an individual basis.

The object of this paper was not only to review a collection of equations. It was also to illustrate the pitfalls that may await the numerical analyst who makes no attempt to analyze the integral equation. Thus, we have displayed an innocuously simple looking Cauchy integro-differential equation, the fluid suction equation of P6, and demonstrated that even though we were able to obtain credible-looking numerical results which appeared to satisfy the asymptotic results, the calculated solution was, in fact, quite wrong: a nonexistence proof makes this clear. If more collocation points are added, the discretization can be seen to break down with the numerical solution displaying ever larger oscillations. Even when asymptotic results are available and the equation has been regularized, designing an appropriate numerical method is not necessarily straightforward. For example, if in problem P8 (Section 6) the collocation points had been chosen at the mesh mid-points (rather than at the mesh points themselves) then this would have led to an ill-posed numerical problem.

Section 6 ends on a more optimistic note. It provides a strategy whereby we might (though we use the term speculatively) obtain a numerical solution to a given Cauchy integral or integro-differential equation. The first step, if the equation contains parameters, is to choose these parameters to simplify the equation, preferably recovering a linear equation. Even if the choice of parameters is not physical, this does not normally matter. The linear equation will then often admit an analytic solution by standard means. If this happens, then so much the better; if it does not, the next step is to attempt to find asymptotic results for both end points of the range. These may be used to regularize the equation which may then admit a numerical solution by traditional means. With the confidence and insight gained from this associated simpler equation we may then return to the original problem. Results from the analysis of the simpler equation and any physical insight that has been gained regarding the underlying equation should then allow one to determine the necessary asymptotics, to regularize and to solve the full problem numerically. Our experience is that this is usually a successful strategy.

Finally, we note that in many ways we have only begun to face the real difficulties of integral and integro-differential equations that involve Cauchy kernels. Cauchy Volterra integral equations (see [**15, 18**]) have not been discussed, and the challenges posed by unsteady problems that involve partial singular integro-differential equations (see, for example [**29**]) are an order of magnitude greater than any of the problems that we have considered above.

Appendix

In this Appendix we list some other examples of LSIDEs and NL-SIDEs that may be used as substitutes for the problems presented in the collection of Section 3. The nomenclature "**An**" is used to signify a problem that is similar to and may be regarded as a companion to problem **Pn** appearing in Section 3. We do not give derivations or many details of each problem below, but instead refer the interested reader to the original publications.

A1. MOSFET/*heat conduction equation.* Many alternatives are available to P1. LSIDEs with a Cauchy kernel posed on a semi-infinite range occur naturally in many flow and heat transfer problems. One example is the "slab heating equation" equation (see [**95**])

$$-\frac{1}{\pi}\,⨍_0^{\infty} \frac{S(t)}{t-x}\,dt = S'(x) \tag{39}$$

with $S(0) = S_0$ which arises in determining the steady cooling of a "slab" (a half space $y < 0$) that lies below a fluid at temperature T_f in $y > 0$. For $x < 0$ the slab/fluid boundary is insulated, while for $x > 0$ Newton cooling with $T_y \propto (T - T_f)$ takes place. In (39), S denotes the quantity $T(x, 0) - T_f$; the boundary conditions are $S(0) = S_0$ and $S(\infty) = 0$. The same equation also arises in the theory of MOSFET devices and in the context of jet-flap theory (see, for example [**69**]). Many closely related examples of LSIDEs occur in other practical problems. These include the "dock" equation (see [**32**]) and its various generalizations; higher order alternatives to (39) are also available (see [**95**]).

One extremely valuable feature of (39) is that, using the methodology outlined in [**95**], closed-form solutions may be determined for comparison purposes. This extremely elegant method can handle equations of the form

$$\frac{1}{\pi}\,⨍_0^\infty \frac{v(t)}{t-x}\,dt = u(x), \quad 0 \le x < \infty,$$

where $u(x)$ and $v(x)$ may be expressed in terms of the unknown function $S(x)$ and its derivatives as

$$u = a_n \frac{d^n S}{dx^n} + a_{n-1} \frac{d^{n-1} S}{dx^{n-1}} + \cdots + a_0 S$$

and

$$v(x) = b_n \frac{d^n S}{dx^n} + b_{n-1} \frac{d^{n-1} S}{dx^{n-1}} + \cdots + b_0 S$$

and the a_i and b_i are constants.

As an example, we note that using this method it can easily be shown that (39) has the closed-form solution

$$S(x) = -\int_0^\infty \ell(s) e^{-xs}\,ds$$

where

$$\ell(s) = -\frac{S_0}{\pi s^{1/2}(1+s^2)^{3/4}} \exp\left(-\frac{1}{\pi}\int_0^s \frac{\log t}{1+t^2}\,dt\right)$$

and $S_0 = S(0)$.

A2. *Geothermal energy equation.* Geothermal power generation, where hot water from as many as five kilometers deep is "mined" to provide power, is a promising ecologically-friendly energy source. The motion of a subterranean one-dimensional fluid-filled partially open crack was considered in [**27**], where the equation

$$⨍_0^\infty \frac{S'(t)}{t-x}\,dt = S(x) + \frac{\lambda}{3}\log\left(\frac{1+x}{x}\right) + \frac{\lambda}{3(1+x)}\log x \tag{40}$$

with $\lambda = (9/2)^{1/3}$ subject to $S(0) = K$, $S(\infty) = 0$ and $S'(0) = 0$ was derived. The problem is to determine $S(x)$ (a measure of the crack height) and the constant K (a physically important quantity). The nonconstant functions on the right-hand side of (40) suggest that it may be a little harder to solve than P2, but most of the methods that work successfully for P2 should also succeed here. Although (40) is first order while the equations in P2 are second order, both problems involve the determination of an eigenvalue of some sort. In [**27**] the equation was first transformed into the finite interval $[-1, 1]$ by setting $x = (1+\xi)/(1-\xi)$ and then solved (using collocation) for various values of K to determine the unique K for which $S'(0) = 0$.

A3 *Grain boundary diffusion.* An interesting (nonconstant coefficient) alternative to the higher order LSIDE of P3 is provided by the equation

$$\text{(41)} \qquad ⨍_{-\infty}^{\infty} \frac{S'''(t)}{t-x}\, dt = xS'(x)$$

subject to the boundary conditions $S(0) = 0$, $S(\pm\infty) = 1$ and $S'(0) = 0$ which was proposed in [**4**] as a model for stress-induced atomic diffusion along a semi-infinite grain boundary. In contrast to the equation of P3, (41) is posed over an infinite interval; though this may introduce some numerical complications, it was shown in [**4**] that a Mellin transform may be used to reduce the equation to a Riemann-Hilbert boundary value problem. The equation may then be solved in closed form, though the solution so derived is extremely unwieldy.

A4 *Eddy breakdown equation*; *flow down a step.* An application of thin aerofoil theory to the nonlinear instability of separated subsonic flow was shown by Brown et al. ([**7**]) to give rise to the NLSIDE

$$\text{(42)} \qquad -\frac{1}{\pi} ⨍_{-d}^{d} \frac{S'(t)}{t-x}\, dt = \frac{1}{8}\left(x^2 - AS^2(x)\right) + B$$

subject to the conditions $S(\pm d) = S'(\pm d) = 0$. Perhaps the simplest way to pose this problem is to assume that d is known, and that A and B must be determined. Thus, four boundary conditions are appropriate.

There are obviously great similarities between this problem and P4; [**7**] did not consider the numerical problem in detail, however.

A modeling refinement of P4 that shares many of its features was proposed in [**79**]. Experimental investigations of high Reynolds number laminar flow down a step suggested that directly downstream of the step a region of constant pressure occurs before the constant-vorticity eddy modeled by P4. Once again, thin aerofoil theory was used to derive the nondimensional NLSIDE

$$\frac{1}{\pi}\int_0^\alpha \frac{S'(t)}{t-x}\,dt = \begin{cases} -A/2 & 0 < x < 1 \\ -A/2 + (S^2(1) - S^2(x)) & 1 < x < \alpha \end{cases} \tag{43}$$

where $S(0) = A/\lambda$, $S(\alpha) = 0$ and $S'(0) = S'(\alpha) = 0$. Here the $O(1)$ constant λ is related to the (known) step height, but four boundary conditions are necessary since the equation is first order and both A (the nondimensional pressure in the constant-pressure region) and the nondimensional reattachment point α are unknown.

A5. *Angled film cooling equation.* In some turbine blade film cooling applications (see P5) it is desirable to preserve the structural integrity of a turbine blade by the addition of a small angled fillet placed at the upstream edge of the injection slot. An NLSIDE for the position of the separating streamline $y = S(x)$ when such a fillet is present was derived in [**30**] in the form

$$\frac{H}{\pi D}\log\left(\frac{x+D}{x}\right) - \frac{1}{\pi}\int_0^\infty \frac{S'(t)}{t-x}\,dt = \begin{cases} -\dfrac{1}{2} & 0 < x < 1 \\ -\dfrac{1}{2} + \dfrac{M^2}{2S^2(x)} & 1 < x < \infty. \end{cases} \tag{44}$$

The boundary conditions are $S(0) = H + d$, $S'(0) = H/D$ and $S(\infty) = M$. H, d and D are given constants that define the shape and orientation of the fillet, and, as in P5, M is the unknown mass flow from the slot that must be determined. This problem is similar to P5 and similar numerical methods may be employed; there are some extra complications however since the presence of the logarithmic term does not allow M to be eliminated from the equation as in P5.

A6. *Waveless ship equation.* An alternative to the nonlinear equation of P6 is provided by studying "waveless" ships. The maritime defense industry has long pursued the holy grail of a waveless (and therefore practically undetectable) ship. A two-dimensional version of this famous problem was studied in [**92**], where equations were derived under a variety of assumptions. The most general form of the waveless ship equation may be written as the Nekrasov equation

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{S(t)}{t-x}\, dt = \frac{1}{3} \log \left[K - 3\gamma \int_{x_0}^{x} \sin S(t) \left(1 + \frac{J/\pi}{t-\beta} \right) dt \right] \tag{45}$$

with K, γ, J, β and x_0 given constants. The unknown function $S(x)$ represents the angle between the streamlines and the x-axis. The parameter J represents the jet flux, and when there is no splashing its value is zero. The parameter K is an arbitrary constant of integration. Important classes of flows have $K = 0$ and $K = 1$. The constant $\gamma = g/U^3$, where g is gravity and U is the free-stream velocity. The parameter x_0 takes the values $x_0 = 0$ when $K = 0$ and $x_0 = \infty$ when $K = 1$. Finally β is a mapping parameter chosen according to the jet flux parameter J, and the single boundary condition required is

$$\int_0^{\infty} \sin S(t)\, dt = -\frac{1}{3\gamma}.$$

For details of experiments that suggest that S may be small, thus allowing linearization and further simplification, the reader is referred to [**70**]. Superficially (45) may seem to have few similarities to the equation examined in P6, for there is no derivative and the region is semi-infinite. The two equations nevertheless share a key similarity, for both the existence and uniqueness of solutions to (45) (and therefore the possible nonexistence or nonuniqueness of waveless ship profiles) depend crucially on the value of the parameter γ.

A7. *Film drainage in droplet coalescence.* An alternative to the nonlinear, infinite range, higher order problem P7 is furnished by the equation

$$\int_{-\infty}^{\infty} \frac{S'(t)}{S^2(t)(t-x)}\, dt = S(x)S'''(x) \tag{46}$$

which was discussed in [**48**]. Here $S(x)$ is a scaled dimensionless variable denoting the gap thickness when a fluid drop settles into a bulk phase. The boundary conditions are $S''(\infty) = \beta$, $S'(-\infty) = 1$ and $S_{\min} = \alpha$; the constants α and β must be determined by solving another problem. No attempt was made by the authors of [**48**] to solve (46) either numerically or asymptotically; they remarked that "There seems to be no theory for equations of this type, and the best advice available to the authors is that at present there is no real prospect of finding a solution numerically."

A8. *Magma migration equation.* An interesting alternative to P8 with a semi-infinite range is given by the equation

$$\left(\frac{1}{\pi}\int_0^\infty \frac{S'(t)}{t-x}\,dt\right)_x = 1 - \frac{1}{S^2(x)} \tag{47}$$

subject to $S'(0) = \infty$, $S(0) = 0$ and $S(\infty) = 1$, which was developed in [**88**]. Here S denotes the width of a buoyancy-driven, magma filled crack in subterranean rock. Once again, the singular integral term is differentiated and it transpires that it is most important to determine the asymptotic properties of the solution. This was duly done in [**88**], where numerical solutions of (47) were also determined using a spectral Chebyshev method.

A9. *Elastohydrodynamic cavity flow.* The extra complications of nonlinearity, higher order, and the fact that the singular integral is itself differentiated are all present in P9. Another equation where such difficulties arise was given in [**86**] in the form

$$\left(S^3(x)\left(\frac{1}{\pi}\int_0^1 \frac{S'(t)}{t-x}\,dt\right)_x\right)_x = \lambda x S'(x) - \beta S(x) \tag{48}$$

with $S(1) = S'(0) = 0$ and $\int_0^1 S(t)\,dt = q$. Here $\beta = \alpha/2 - 1/6$, $\lambda = \alpha/2 + 1/6$ and q and α are known constants: $S(x)$ denotes the similarity form of the width of an elastohydrodynamic lens-shaped cavity lying in a plane of cleavage between two elastic half-spaces that is filling with a viscous fluid. Asymptotic analysis was carried out on (48) to determine the behavior of the solution near to the crack tips and a numerical solution was calculated using a Chebyshev spectral method applied to a much-transformed version of (48), once again emphasizing the theme that it may often be better to carry out a considerable amount of work on the NLSIDE before attempting a solution.

A10. *Properties of dilute magnetic alloys.* Though alternatives to P10 are not easily found, they do exist. In [**68**] a model of a dilute magnetic alloy with one localized impurity spin led to the two coupled NLSDEs

$$
\begin{aligned}
&f(x) + h(x)g(x) = n(x) + \frac{J}{2} \fint_{-D}^{D} \frac{\rho(t)g(t)}{t-x}\,dt,\\
&g(x)\left(1 - h(x) + J \fint_{-D}^{D} \frac{\rho(t)f(t)}{t-x}\,dt\right)\\
&\qquad + f(x)\left(S(S+1)h(x) - J \fint_{-D}^{D} \frac{\rho(t)g(t)}{t-x}\,dt\right)\\
&\qquad = \frac{1}{2} S(S+1) J \fint_{-D}^{D} \frac{\rho(t)f(t)}{t-x}\,dt - \frac{J}{2} \fint_{-D}^{D} \frac{\rho(t)g(t)}{t-x}\,dt.
\end{aligned}
$$

Here the unknown functions (components of the "t"-matrix) are $f(x)$ and $g(x)$, S is the impurity spin magnitude, D is the conduction band semi-width, J is the anti-ferromagnetic coupling and $\rho(x)$, $n(x)$ and $h(x)$ are known functions connected to the Fermi-Dirac distribution function and the density of states for the conduction band. A numerical solution of these equations was not attempted, for, as shown in [**68**], the particular structure of the associated Riemann-Hilbert problem allows an exact solution to be determined.

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