Surv. Math. Ind. (1996) 6: 21–53

Surveys on Mathematics for Industry

Mixed systems of conservation laws in industrial mathematical modelling

Alistair D. Fitt, Southampton

Dedicated to the memory of Alan Tayler

Summary. Many mathematical models of evolutionary industrial processes may be written as $N \times N$ systems of conservation laws in terms of N independent variables comprising time and $N-1$ space variables. If such systems possess real and distinct eigenvalues, they are said to be strictly hyperbolic. For 'mixed systems', however, the eigenvalues may be equal at points in phase space or even fail to be real, so that the problem has both hyperbolic and elliptic characteristics. In this case the system is ill-posed and requires the specification of boundary conditions that can violate causality. Mathematical models of physical processes that lead to mixed equations are discussed and reviewed, and some of the properties of mixed systems are compared to those of hyperbolic systems. The significance of prototype systems that have been proposed specifically to analyse such properties is considered, and attention is then turned to the archetypal mixed system; the two-phase flow equations. Possible resolutions of the two-phase flow dilemma are compared, and a manner in which the modelling may be approached via a more general rational asymptotic scheme is indicated.

AMS Subject Classification: 35M10, 35L67, 76T05.

Keywords: Mixed elliptic/hyperbolic systems of conservation laws; Two-phase flow; Riemann problems; Numerical methods for conservation laws; Mathematical modelling of industrial processes.

1 Introduction

Systems of conservation laws of mixed type occur in a wide variety of mathematical modelling problems. "Mixed" systems are distinguished by the fact that they possess both real and complex eigenvalues, unlike the more familiar hyperbolic, parabolic or elliptic systems of conservation laws where the eigenvalues are either pure real or pure imaginary. This survey discusses the types of models that can give rise to mixed systems, giving particular attention to mixed systems that occur in industrial contexts. A variety of models will be discussed more fully in the subsequent sections, but two-phase flow problems will receive the largest amount of attention; such flows are so common in a wide variety of industrial processes and there has been so much attendant confusion and controversy that there seems to be no consensus on how to proceed in general.

To practitioners that have not encountered them before, the appearance of mixed systems is often quite unexpected. The strictures of profit and production dictate that in most cases industry cannot wait for a complete academic resolution of the multitude of complicated practical problems that are posed by mixed systems. For this reason this survey concentrates not only on the mathematical technicalities of mixed problems, but also addresses the matter of how practical and sometimes "dirty" techniques may be used to determine important results. We shall see below that it is probably possible to construct cogent mathematical arguments for completely discarding many mixed systems that arise from models of industrial processes. Once again, however, the practicalities of industrial development render the understanding of such systems wholly worthwhile, as in many cases the existing model, though it leads to mixed equations, may be the *only* available model and must therefore be used in spite of its shortcomings. Clearly in these circumstances any indication of how reliable the results that are produced might be is of the greatest value.

In Sect. 2 some notation is standardized and some mathematical preliminaries are dealt with. Some of the more standard properties of scalar equations and hyperbolic systems are then highlighted so that the differences and similarities in behaviour when the conservation laws are of mixed type can be stressed. In Sect. 3 a selection of mathematical models are introduced; all were originally developed to attempt to explain physical processes, many are relevant to modern industry and all give rise, in certain circumstances, to mixed systems of conservation laws. Section 4 summarizes the similarities and differences between these models and discusses the effect that the mixed nature of the equations has exerted on model development.

Much effort has been expended in recent years on the determination of the properties of specially constructed prototype mixed systems. Attention has been given not only to systems with both real and imaginary eigenvalues, but also to "nonstrictly hyperbolic" systems where, though the eigenvalues are real, they may coalesce, leading to solutions with novel properties. Progress is reviewed in Sect. 5. In Sect. 6 we turn in more detail to the specific problem of two-phase flow, considering various different approaches that have been used to deal with the mixed nature of the problem. Numerical methods are discussed, and, proceeding upon technical grounds alone, it soon becomes evident that numerically the outlook is somewhat bleak. It is possible to take a more pragmatic view however, and some examples are given of circumstances where useful numerical results may be produced. Finally, in Sect. 7 some conclusions are drawn regarding mixed problems and some issues are identified that need to be addressed if the theory is to be advanced.

2 Mathematical preliminaries

To standardize notation, it is assumed that we wish to investigate the properties of an $N \times N$ first-order system of conservation laws of the form

$$A\mathbf{w}_{x_1} + B\mathbf{w}_{x_2} = \mathbf{b} \tag{1}$$

Here $\mathbf{w}$ is a vector composed of the N unknowns (dependent variables) in the problem, A and B are $N \times N$ matrices (which in general will depend on $\mathbf{w}$ as well as x_1 and x_2) and $\mathbf{b}$ is the "source term" vector, which may depend on x_1, x_2 and $\mathbf{w}$, but contains no derivatives of the independent variables. Typically (though not always), x_1 will be taken to be the time t and x_2 will represent a distance x. If this is the case then it will be assumed throughout (unless otherwise stated) that t and x have been identified respectively as "timelike" and "spacelike" variables and suitable initial and boundary conditions have been specified. (It will become apparent that the specification of "suitable" boundary conditions for mixed systems may be a far from trivial problem.) As usual, the eigenvalues $\lambda = \mathrm{d}x_2/\mathrm{d}x_1$ of the system (1) are defined to be solutions of the characteristic equation

$$\det(B - \lambda A) = 0,$$

whilst the (right) eigenvectors are vectors $\mathbf{x}$ that satisfy

$$(B - \lambda A)\mathbf{x} = 0.$$

The eigenvalues and eigenvectors lead directly to the Pfaffian differential equations for the system, confirming that, along characteristics, the system reduces to ordinary differential equations. In some cases the Pfaffian equations may be solved, thus determining the Riemann invariants of the system.

In general, the eigenvalues are the solutions of an N-th degree polynomial equation whose coefficients are functions of $\mathbf{w}$, x_1 and x_2. We refer to the system (1) as *hyperbolic* if all the eigenvalues are real and *mixed* if some are real and some are complex with non-zero imaginary parts. From a rigorous point of view, the definition of hyperbolicity given above is a little loose, for the normal requirement for strict hyperbolicity is that the eigenvalues are real and distinct. For the conservation laws that will form the main body of our study, however, the eigenvalues depend on the elements of $\mathbf{w}$ as well as x and t and there may be regions in phase space where strict hyperbolicity does not hold. This can happen in various ways, but we group all such systems together by referring to them as *nonstrictly hyperbolic*. In cases where the eigenvalues are real but identically equal, we refer to (1) as *parabolic*. Even if a system is strictly hyperbolic, the properties of the eigenvalues may greatly influence the sort of discontinuities that might be expected to appear in solutions. Given an eigenvalue λ_k and a corresponding right eigenvector $\mathbf{x}_k$, we define λ_k to be *genuinely nonlinear* if $\nabla\lambda_k \cdot \mathbf{x}_k \neq 0$. If every eigenvalue is genuinely nonlinear, then we refer to (1) as a *genuinely nonlinear system*. Frequently in the study of mixed systems we encounter eigenvalues that are genuinely nonlinear except at a number of isolated points in phase space where $\nabla\lambda_k \cdot \mathbf{x}_k = 0$. Such points are referred to as *fognals*.

Although the definitions given above refer to $N \times N$ systems, our major concern will frequently be the study of mixed systems of two equations in two unknowns with two independent variables. Such a restriction enjoys the great advantage that it is possible to study solutions by examining two-dimensional phase space.

2.1 *A review of some relevant theory for hyperbolic equations*

The single conservation law

To fix ideas, consider the single conservation law

$$u_t + f(u)_x = 0; \quad u(x, 0) = u_0(x). \tag{2}$$

It is well-known that solutions to such equations can develop discontinuities (shock solutions) in finite time. In the case of mixed systems we will naturally be interested in shock-type solutions, so what can be said about solutions to (2)? Firstly, we must decide what we mean by the term "solution", for the equation contains derivatives and clearly these cannot be defined in the standard way for discontinuous solutions. Suppose that we are interested in solutions of (2) which are defined for $a \le x \le b$ and $0 \le t \le \tau$ Henceforth we regard a "solution" to (2) as a (possibly discontinuous) bounded measurable function $u(x, t)$ with bounded measurable initial data u_0 which satisfies

$$\iint_{t \ge 0} (u\psi_t + f\psi_x)\,\mathrm{d}A + \int_{t=0} u_0\psi\,\mathrm{d}x = 0 \tag{3}$$

for all test functions ψ with continuous first derivatives and compact support on a suitable region, for example, $D = \{(x, t) : a \le x \le b, 0 \le t \le \tau\}$. The advantage of this definition of a "weak" solution is obvious; all continuous ("classical") solutions are weak solutions (a fact easily confirmed by applying Green's theorem to (2)), but the condition (3) does *not* require $u(x, t)$ to be continuous.

The definition (3) allows $u(x, t)$ to possess only very special types of discontinuity, for suppose Γ is a curve across which $u(x, t)$ is discontinuous and which separates D into two regions D_1 and D_2 with $D = D_1 \cup D_2$. Assuming that $u(x, t)$ has continuous derivatives in each of D_1 and D_2, an application of Green's theorem in D_1 and D_2 separately shows that

$$\int_\Gamma \psi([f(u)]\,\mathrm{d}t - [u]\,\mathrm{d}x) = 0, \tag{4}$$

the square brackets denoting the jumps in u and $f(u)$ as the boundary Γ is crossed. The arbitrary nature of ψ means that (4) can only be true in general if the shock speed $s = \mathrm{d}x/\mathrm{d}t$ satisfies the jump condition (sometimes termed a Rankine–Hugoniot condition)

$$s = \frac{[f(u)]}{[u]}. \tag{5}$$

It has long been known that the definition of a weak solution and the satisfaction of the jump condition is not enough for existence and uniqueness. A further condition, usually referred to as an *entropy* condition is required, and for the single conservation law this asserts that, if u_L and u_R are the values of u respectively to the left and right of a jump in the solution, then

$$f'(u_R) < s < f'(u_L). \tag{6}$$

There are many compelling physical reasons why (6) should hold. The condition requires that the shock speed lies between the characteristic speeds on either side of the shock, and is closely related to the existence of a viscous profile (see below). It also plays a crucial role (see, e.g., [45]) in the proof of existence and uniqueness of solution for the scalar conservation law. More generally, the entropy condition may be thought of as being an *admissibility* condition that selects the most physically reasonable solution from a number of candidates.

The definition of a weak solution, together with the notions of jump and entropy conditions provide all the required elements to solve problems such as (2) by piecing together shocks with continuous regions of the flow.

2.2 *The Riemann problem for hyperbolic systems of conservation laws*

The mathematical models that will be considered below inevitably involve systems of conservation laws. A particularly useful special case that is frequently studied for simplicity occurs when the initial data $\mathbf{w}_0$ for a system is composed only of two constant states, say $\mathbf{w}_L$ and $\mathbf{w}_R$, separated by a jump at $x = 0$. This is generally known as a *Riemann problem.*

Although much is known about the scalar conservation law considered above, even for source-free $N \times N$ hyperbolic systems of the form

$$\mathbf{w}_t + \mathbf{f}(\mathbf{w})_x = 0, \quad \mathbf{w}(\mathbf{x}, 0) = \mathbf{w}_0(\mathbf{x}), \tag{7}$$

(where $\mathbf{w} = (w_1, w_2, \ldots, w_N)$) much less theory is available. The most important existence theorem is due to Glimm [12]. This asserts that, if the eigenvalues and eigenvectors of (7) are denoted respectively by λ_k and $\mathbf{z}_k$, then if the system is hyperbolic (all the λ_k real and distinct) and genuinely nonlinear ($\nabla\lambda_k \cdot \mathbf{z}_k \neq 0$ for all k) in some open subset of $\mathscr{R}^N$, existence of the solution is guaranteed for all $t > 0$ *provided* that the total variation of $\mathbf{w}_0$ is sufficiently small.

The restriction on the size of the initial data arises from the fact that the proof of Glimm's theorem relies essentially on showing that approximate solutions formed by joining together the solutions of Riemann problems possess a limit which is a solution; this is so only if the initial states are sufficiently close to each other. For particular systems with less restrictive initial data there are some existence results, (see, for e.g., [30]) but the general problem remains unsolved, as does the question of uniqueness.

Naturally, for (7) the Rankine–Hugoniot and entropy conditions (5) and (6) must be generalized. This may be accomplished in a natural way by requiring that

$$s(\mathbf{w}_L - \mathbf{w}_R) = \mathbf{f}(\mathbf{w}_L) - \mathbf{f}(\mathbf{w}_R) \tag{8}$$

(thereby replacing (5) with N equations for $\mathbf{w}_L$, $\mathbf{w}_R$ and s), and denoting solutions as permissible in the entropy sense so long as for some k

$$\lambda_k(\mathbf{w}_R) < s < \lambda_{k+1}(\mathbf{w}_R), \quad \lambda_{k-1}(\mathbf{w}_L) < s < \lambda_k(\mathbf{w}_L). \tag{9}$$

The latter condition has come to be known as the *Lax* entropy condition for a k-shock. Assuming that the eigenvalues are ordered in increasing order of magnitude, it is equivalent to the statement that, for a k-shock,

$$\lambda_k(\mathbf{w}_R) < s < \lambda_k(\mathbf{w}_L) \tag{10}$$

As far as the Riemann problem for strictly hyperbolic systems of order N is concerned, a solution may generally be constructed in terms of a collection of N centred waves, which may be either shocks or rarefactions. When N exceeds 2, contact discontinuities (arising from a failure of the system to be genuinely nonlinear) may also occur; these phenomena are familiar, for example, to gas dynamicists.

2.3 2×2 *Hyperbolic systems of conservation laws*

As discussed above, $N \times N$ hyperbolic systems of a general nature are normally hard to analyze. When $N = 2$, however, the Riemann problem is well understood. The Riemann problem was solved ([46], [47]) for genuinely nonlinear 2×2 hyperbolic systems, showing uniqueness for the case where the solution was required to satisfy the Lax entropy inequalities. The work of Liu [26] showed that the Oleinik entropy condition for a single equation could be extended in a natural way to encompass 2×2 hyperbolic systems when the condition of genuine nonlinearity was relaxed. As far as the 2×2 hyperbolic Riemann problem is concerned, the general structure of the solution is well established, and consists of constant states joined by a maximum of two waves, which may be either shocks or rarefactions.

2.4 *The viscous profile*

One of the central issues concerning the study of mixed systems of conservation laws is embodied in the idea of a *viscous profile*. The strictly hyperbolic 2×2 system

$$\mathbf{w}_t + \mathbf{f}(\mathbf{w})_x = 0 \tag{11}$$

is "inviscid" in the sense that it contains no (dissipative) second derivative terms. It is common experience that, for example, in gas dynamics, typical shocks have a width of only a few mean free paths. For this reason it is normal to ignore dissipative terms such as viscosity and admit jumps in the solution. Although this approximation is adequate for all practical purposes, it is understood that the inviscid solution arises, in some carefully-defined mathematical sense, from a fully viscous problem. Suppose that a shock solution $(\mathbf{w}_L, \mathbf{w}_R; s)$ to Eq. (11) connects a left state L to a right state R with a shock of speed s that satisfies

$$s(\mathbf{w}_L - \mathbf{w}_R) = \mathbf{f}(\mathbf{w}_L) - \mathbf{f}(\mathbf{w}_R)$$

and the standard entropy conditions. Is it then possible to "determine the structure in the shock" by adding a "viscous" term to the right-hand side of (11)? Evidently if a solution of

$$\mathbf{w}_t + \mathbf{f}(\mathbf{w})_x = \varepsilon \mathbf{w}_{xx} \tag{12}$$

can be found in the form of a travelling wave $\mathbf{w} = \mathbf{w}((x - st)/\varepsilon)$ which tends to $(\mathbf{w}_L, \mathbf{w}_R; s)$ as $\varepsilon \to 0$, then the inviscid solution may be regarded as "valid". Inserting the travelling wave into (12), we find, after an integration, that

$$-s\mathbf{w} + \mathbf{f}(\mathbf{w}) = \mathbf{w}' - C,$$

the constant of integration being given in terms of $\mathbf{w}_L$ (or equivalently, because of the jump conditions, in terms of $\mathbf{w}_R$) by

$$C = s\mathbf{w}_L - \mathbf{f}(\mathbf{w}_L)$$

The problem has therefore been reduced to solving the ordinary differential equations

$$\mathbf{w}' = \mathbf{V}(\mathbf{w}), \quad \mathbf{V}(\mathbf{w}) = -s(\mathbf{w} - \mathbf{w}_L) + \mathbf{f}(\mathbf{w}) - \mathbf{f}(\mathbf{w}_L)$$

subject to suitable boundary conditions as $\varepsilon \to 0$. Topological methods may be employed ([45]) to prove existence of the viscous profile *provided* the quantity $|\mathbf{w}_L - \mathbf{w}_R|$ is sufficiently small; put another way, shock solutions of hyperbolic systems are guaranteed to possess a viscous profile provided the shock is weak enough.

We shall see that, on many occasions, mixed problems lead to novel types of shocks that are not present in hyperbolic systems. The existence of a viscous profile in such cases forms a key piece of evidence that these unconventional discontinuities deserve the status of physically meaningful results, rather than mere synthetic mathematical inventions.

3 Mixed systems of conservation laws in industrial problems

Systems of conservation laws which contain elliptic regions of phase space occur in the modelling of many processes that are of interest to industry. In this section we identify a selection of such cases and explain the model formulation that leads to a mixed system.

3.1 *Transonic flow*

The study of transonic flow gives rise to what may be thought of as the classical mixed problem. Consider first the case where the two-dimensional steady inviscid flow of a compressible gas past an aerofoil is to be determined. Denoting the velocity by $\mathbf{q} = (u, v)$, the speed by q and the local sound speed for a given pressure p and density ρ by $a^2 = \mathrm{d}p/\mathrm{d}\rho$, what will be the qualitative details of the flow for a free-stream Mach number $M_\infty = q_\infty/a_\infty$ of approximate size 0.6–0.9? It is a matter of experience (which may, under ideal conditions, be confirmed simply by watching the wings of a commercial jet airliner during level flight) that whilst the majority of the flow around the aerofoil is subsonic ($M < 1$), there exist small regions (typically above and below the more central regions of the aerofoil) where M exceeds 1 and the flow is therefore supersonic. Inevitably, therefore, we are led to the consideration of a *mixed* problem where the equations change type from elliptic to hyperbolic when certain regions (whose locations are unknown at the outset) in the (x, y)-plane are entered.

The reason for the mixed nature of the problem is easily seen when the equations are analyzed, for, writing the continuity, momentum, and energy equations as

$$(\rho u)_x + (\rho v)_y = 0,$$

$$uu_x + vu_y = -\frac{1}{\rho} p_x,$$

$$uv_x + vv_y = -\frac{1}{\rho} p_y,$$

$$\left(\frac{1}{2}\rho u(u^2+v^2) + \frac{up\gamma}{\gamma-1}\right)_x + \left(\frac{1}{2}\rho v(u^2+v^2) + \frac{vp\gamma}{\gamma-1}\right)_y = 0,$$

and assuming the perfect gas law so that $p = \rho RT$ where $R = c_p - c_v$, it is a simple matter to express the equations in the form (1) and determine the eigenvalues. This gives

$$\lambda = \frac{v}{u} \quad \text{(twice)}$$

and

$$\lambda = \frac{uv \pm c\sqrt{q^2 - c^2}}{u^2 - a^2},$$

and it is immediately obvious that two of the eigenvalues are real for supersonic flow and complex for subsonic flow. We note in passing that there are other formulations of the transonic flow problem such as the TSD (transonic small disturbance) model where the aerofoil is assumed to be slender and the flow is predominantly in one direction and the full potential model where we assume that the density is a given function of the velocity Even when such simplifications are applied however, the mixed nature of the equations still persists.

Since the transonic equations are elliptic only in two space, rather than one space and one time variable, they will not be studied in great detail below. It is worth mentioning, however, that systems that are mixed in space only occur also in other physical problems, typical examples being plasma modelling [55] and granular flow [54].

3 2 *Models for traffic flow*

Chronologically, probably the first modelling study apart from transonic flow where conservation laws of mixed type were encountered was that carried out by Bick and Newell [3] who considered the case of bi-directional traffic flow on an undivided highway Denoting the densities of vehicles in the right- and left-hand lanes of the highway by p and q respectively, and assuming that associated average velocities in the lanes are given by $U \geq 0$ and $V \geq 0$, then a simple "conservation of cars" argument leads to the system

$$p_t + (pU)_x = 0\,, \quad q_t + (qV)_x = 0\,. \tag{13}$$

This system of two equations in four unknowns must somehow be closed. For unidirectional traffic it has long been known that for many practical traffic flow scenarios the linear relationship $U = U_0 - \alpha p$, where α is constant, gives reliable results. An obvious generalization to this constitutive law may be proposed for the bi-directional highway by assuming that

$$U = V_0 - \alpha p - \beta q\,, \quad V = -U_0 + \alpha q + \beta p \tag{14}$$

where $\alpha > \beta > 0$ were taken to be constant. (Amongst other desirable properties, this choice ensures that $U(p, q) = -V(q, p)$ so that both lanes are physically identical.) After eliminating U and V, (13) may be written in standard conservation law form, and we find that the roots are real whenever

$$(p^2 + q^2)(2\alpha + \beta)^2 + 2pq(4\alpha^2 - \beta^2 + 4\alpha\beta) + (U_0 + V_0)^2$$
$$- 2(p + q)(\beta + 2\alpha)(U_0 + V_0) \geq 0$$

This represents the region outside an ellipse in phase (p, q) space which may be conveniently written

$$C\bar{P}^2 + D\bar{Q}^2 \geq E$$

where

$$\bar{P} = \left(p + q - \frac{(\beta + 2\alpha)(U_0 + V_0)}{4\alpha(\alpha + \beta)}\right)/\sqrt{2}, \quad \bar{Q} = (q - p)/\sqrt{2},$$

and

$$C = 8\alpha(\alpha + \beta), \quad D = 2\beta^2\,, \quad E = \frac{\beta^2(U_0 + V_0)^2}{4\alpha(\alpha + \beta)}.$$

The relevant ellipse therefore has major and minor axes of length $(U_0 + V_0)\beta/(\sqrt{8}\alpha(\alpha + \beta))$ and $(U_0 + V_0)/\sqrt{2\alpha(\alpha + \beta)}$ respectively

3.3 One-dimensional unsteady flow of a van der Waals' gas

One of the most frequent explanations given for the existence of a mixed conservation law system is that it has arisen from a complicated and poorly understood model. The next example illustrates that mixed systems may arise from the most well-established of models when ostensibly reasonable changes are made to the constitutive laws.

We consider the one-dimensional unsteady flow of a compressible gas of density ρ, velocity u and pressure p. Suppose that the absolute temperature T_G of the gas is constant (isothermal flow) and the relationship between p and ρ is assumed known. The equations of motion are

$$\rho_t + (\rho u)_x = 0, \tag{15}$$

$$u_t + uu_x + \frac{p_x}{\rho} = 0 \tag{16}$$

which may be regarded as two equations for the two unknowns ρ and u. It is easily confirmed that the eigenvalues of this system of conservation laws are given by

$$\lambda = u \pm \sqrt{\frac{\mathrm{d}p}{\mathrm{d}\rho}}.$$

For the familiar case of an ideal gas, we have $p(\rho) = R\rho T_G$ where R is a universal constant, and the conservation laws are therefore hyperbolic. More generally, it is clear that hyperbolicity is assured so long as the 'gas law' is of such a form that $\mathrm{d}p/\mathrm{d}\rho > 0$ for all values of ρ under consideration.

It is known, however, from thermodynamic considerations that the ideal gas is, as the name suggests, very much an idealization as far as the real behaviour of gases is concerned; theoretically it provides an accurate description only in the case where the gas in question is composed of infinitely small molecules that behave like perfectly elastic bodies upon impact. More accurate constitutive laws are available, and an example of one such law is due to van der Waals who proposed an equation of state which not only allowed for the fact that the volume of a molecule is finite, but also took into account the presence of attractive and repulsive intermolecular forces. Writing the van der Waals law as

$$p(\rho) = \frac{RT_G\rho}{1 - b\rho} - a\rho^2, \tag{17}$$

where a and b are constants, (b is normally termed the *covolume* constant) we have

$$\frac{\mathrm{d}p}{\mathrm{d}\rho} = \frac{RT_G}{(1 - b\rho)^2} - 2a\rho.$$

For low enough temperatures the pressure/density curve is therefore non-monotonic, and there will be an elliptic region in phase space.

We note in passing that Eqs. (15) and (16) may be written in a much simpler way as the system

$$v_t - u_x = 0, \tag{18}$$

$$u_t - \sigma(u)_x = 0 \tag{19}$$

To prove this, set $v = -\log\rho$ and identify t with the standard convective derivative $\partial_t + u\partial_x$. We then retrieve (15) and (16) with $p_x = \rho_x\sigma'(-\log\rho)$.

Fuller comments on the presence of the elliptic region will be made in due course, but it is worth noting that in an experiment it would prove difficult to ensure that changes in pressure were compensated for only by changes in density. Moreover, we must distinguish carefully between temperatures for which the gas is a true vapour and those for which it becomes a liquid. Is it possible therefore, that the system has proved to be of mixed type simply because the concept of an isothermal van der Waals gas is not physically very realistic?

Evidence that this is not the case may be gathered by looking at the non-isothermal case. The three equations of motion are

$$\rho_t + u\rho_x + \rho u_x = 0,$$

$$u_t + uu_x + \frac{p_x}{\rho} = 0,$$

$$(\rho E)_t + [u(\rho E + p)]_x = 0$$

where $E = e + \frac{1}{2}u^2$ and e is the internal energy As well as the definition of p given by (17), we are now required to define e as a function of T and ρ and for a van der Waals gas the relevant relationship takes the form

$$e = c_v T_G - a\rho.$$

It is easily confirmed that the eigenvalues of the system are given by $\lambda = u$ and

$$\lambda = u \pm \sqrt{-2\rho a + \gamma\left(\frac{\rho a + p/\rho}{1 - b\rho}\right)} = u \pm \sqrt{\frac{\mathrm{d}p}{\mathrm{d}\rho}},$$

and it is evident that adding the effect of non-isothermality has served only to introduce an extra eigenvalue $\lambda = u$. Apart from this change the system still exhibits the same mixed character as the original model.

3 4 *Phase boundaries in elastic bars*

The propagation of phase boundaries in elastic bars was considered by James [22]. Such phase boundaries (which have commonly been observed in natural rubbers, polymers, various metals and other more exotic substances) occur when the stress-deformation relation for a one-dimensional elastic material is not monotone, and resemble shock waves. If an elastic bar is described by a single Lagrangian co-ordinate X, $-L < X < L$, and displacements of the bar are functions $y(X, t)$, then, describing the stress that corresponds to a deformation u by $p(u)$, the equation of motion of the bar is evidently given by

$$y_{tt} = (p(y_X))_X \tag{20}$$

Setting $u = y_X$ and $v = y_t$, (20) may be rewritten as the first-order system of conservation laws

$$u_t - v_X = 0, \tag{21}$$

$$v_t - (p(u))_X = 0. \tag{22}$$

We note that, with minor changes of notation, (21) and (22) are identical to (18) and (19). For this reason, this problem and the van der Waals flow described above have traditionally been bracketed together, and similar remarks apply concerning the existence of elliptic regions of phase space when the constitutive law is non-monotone.

3.5 *Oil recovery*

The presence of complex eigenvalues has caused major problems in the modelling and understanding of the fluid flow occurring in petroleum reservoirs. A revealing study of the equations concerned and properties of solutions to these equations was presented by Bell et al. [2]. The most popular model for flow in such reservoirs assumes that there are three phases present; oil, water and air. These components are assumed to flow within a porous medium, so that D'Arcy's law is applicable. Considering only one-dimensional flow and ignoring the effects of gravity, compressibility and inhomogeneity, the three phases are assumed to completely fill the available pore space. Following Bell et al. we denote the phase saturations by s_w, s_o and s_g (the subscripts referring to water, oil, and gas, respectively) and the phase velocities by v_i ($i =$ w, o, or g). If the medium has porosity ϕ and the pressure of each phase is p_i, then using conservation of mass in each phase and employing the D'Arcy law in the form

$$v_i = -\frac{Kk_{ri}}{\mu_i}\frac{\partial p_i}{\partial x}$$

where μ_i is the phase viscosity, K is the absolute permeability of the medium and k_{ri} is the relative permeability of each phase, it is possible to use the relationship

$$s_w + s_o + s_g = 1$$

to eliminate one of the phase saturations. Removing s_o gives the system of conservation laws

$$\mathbf{s}_t + A\mathbf{s}_x = 0 \tag{23}$$

where $\mathbf{s} = (s_w, s_g)^T$ There is one further assumption which is necessary to derive (23) that has so far not been mentioned, but is worthy of special note. This concerns the phasic pressures. Obviously some law connecting the pressures must be postulated to close the system, and in (23) it has been assumed that the effects of interfacial tension may be ignored, so that the phasic pressures are all equal. It will become evident during the discussion below of models for two-phase flow that the "equal pressures" supposition is a recurring theme in mixed problems.

The matrix A appearing in (23) has a particular form; defining $\lambda_i = k_{ri}/\mu_i$, we find that

$$A = \frac{1}{(\lambda_w + \lambda_o + \lambda_g)^2}\begin{pmatrix} (\lambda_g + \lambda_o)\lambda'_w - \lambda_w\lambda_{o,w} & -\lambda_w(\lambda'_g + \lambda_{o,g}) \\ -\lambda_g(\lambda'_w + \lambda_{o,w}) & (\lambda_w + \lambda_o)\lambda'_g - \lambda_g\lambda_{o,g} \end{pmatrix}$$

Here primes denote differentiation, whilst $\lambda_{o,w} = \partial\lambda_o/\partial s_w$ and $\lambda_{o,g} = \partial\lambda_o/\partial s_g$; the assumption has also been made (based on "reliable experimental data") that k_{rw} and k_{rg} (and therefore λ_w and λ_g) depend only upon s_w and s_g respectively, whilst λ_o is a function of both s_g and s_w.

The model must now be closed by specifying the relative permeabilities. Bell et al. considered the particular case where

$$k_{ro} = \frac{(1 - s_w - s_g)k_{row}k_{rog}}{(1 - s_w)(1 - s_g)}$$

with

$$k_w = 1.09 s_w^{1.5516687} - 0.09 s_w^{4.51668}, \quad k_g = 0.525 s_g^{1.02} + 0.475 s_g^{3.62}$$

and

$$k_{row} = 1.95(1 - s_w)^{8.28} - 0.95(1 - s_w)^{11.284},$$

$$k_{rog} = 1.19(1 - s_g)^{2.006} - 0.19(1 - s_g)^{2.024},$$

these values having been derived from experimental correlations. A standard characteristic analysis shows that a closed elliptic region is present in (s_w, s_g)-space. This should not be regarded as having arisen from a particularly unwise choice of relative permeability model, since for many other experimental correlations elliptic regions also exist. We note, however, that essentially the mixed nature of the problem arises from a "closure law", a recurrent theme in mixed problems.

3.6 Shear-thinning flow

An example of a system of equations that is normally elliptic, but becomes hyperbolic in a certain limit, is furnished by the equations of inertia-free shear-thinning flow. Assuming that such fluids have a stress tensor T_{ij} of the form

$$T_{ij} = -p\delta_{ij} + \tau_{ij}$$

where

$$\tau_{ij} = \mu K \dot{\gamma}_{ij}, \quad K = |(\dot{\gamma}_{kl}\dot{\gamma}_{kl})|^{\frac{\varepsilon - 1}{2}}$$

and p denotes pressure, (u, v) are the velocity components, μ is the (constant) "consistency index", $\dot{\gamma}_{kl}$ is infinitesimal rate-of-strain tensor, and ε is the "shear-thinning" parameter, the non-dimensional equations of motion are

$$-p_x + (Ku_x)_x + \tfrac{1}{2}\,(K(u_y + v_x))_y = 0, \tag{24}$$

$$-p_y + (Kv_y)_y + \tfrac{1}{2}\,(K(u_y + v_x))_x = 0, \tag{25}$$

$$u_x + v_y = 0 \tag{26}$$

Determining the characteristics $\lambda = \mathrm{d}y/\mathrm{d}x$ in the normal way, we find that

$$(\lambda^4 + 1)(\varepsilon + 4R^2) + 8R(\lambda^3 - \lambda)(\varepsilon - 1) + \lambda^2(8R^2(2\varepsilon - 1) + 4 - 2\varepsilon) = 0$$

where $R = u_x/(u_y + v_x)$. For the case $\varepsilon = 1$, corresponding to Newtonian flow, the eigenvalues are $\pm i$, $\pm i$ and the system (being, of course, the standard slow flow equations) is elliptic. It is also easy to show that for any $\varepsilon > 0$ every eigenvalue is complex. For certain types of clay, however, the parameter ε is close to zero, and it may be shown in these cases [4] that boundary layers exist close to any solid walls in the flow. To determine the "outer" solution, ε must be set to zero, in which case the eigenvalues take the (repeated) purely real values

$$\lambda = \frac{1 \pm \sqrt{1+4R^2}}{2R}.$$

In fact, a closer examination of the equations in the case $\varepsilon = 0$ shows that the system may be written as two decoupled hyperbolic equations.

3.7 *Two-phase flow*

Of all the industrial problems that have spawned mixed systems of conservation laws, two-phase flow is probably the most studied and the most controversial. Although Sect. 6 will address the problem of two-phase flow modelling in more detail, a simple model is examined here for the purposes of completeness.

Consider an unsteady one-dimensional two-phase flow where the continuous phase is denoted by the subscript 1 and the discrete phase by a subscript 2. Using ρ, u and p to denote density, velocity and pressure respectively, and employing α to denote the void fraction ($\alpha_1 + \alpha_2 = 1$), a popular model for inviscid incompressible flow with no interphase transfer (see, e.g., [13]) is given by

$$(\rho_1\alpha_1)_t + (\rho_1\alpha_1 u_1)_x = 0, \tag{27}$$

$$(\rho_2\alpha_2)_t + (\rho_2\alpha_2 u_2)_x = 0, \tag{28}$$

$$(\rho_1\alpha_1 u_1)_t + (\rho_1\alpha_1 u_1^2)_x + \alpha_1 p_x = 0, \tag{29}$$

$$(\rho_2\alpha_2 u_2)_t + (\rho_2\alpha_2 u_2^2)_x + \alpha_2 p_x = 0. \tag{30}$$

Since the flow is assumed to be incompressible, this represents a system of four partial differential equations for the four unknowns α_1, u_1, u_2, and p. It will be noted that, in order to close the system, it has been assumed that a single pressure p suffices to describe the pressure in both phases. It becomes clear, however, that the model is defective, for a standard calculation shows that the characteristics are given by $\lambda = 0$ (twice; corresponding to incompressibility) and

$$\frac{\alpha_2}{\rho_2}(u_1 - \lambda)^2 + \frac{\alpha_1}{\rho_1}(u_2 - \lambda)^2 = 0. \tag{31}$$

If $u_1 = u_2$ (a rather trivial case) then clearly all the eigenvalues are zero, but in all cases where the velocities differ obviously the solutions to (31) are complex.

3.8 *Pursuing biological populations*

Suppose that $U(x,t)$ and $V(x,t)$ are the densities of two biological populations, and that the species U is being pursued by the species V along a straight line course. It is assumed that one population encourages the other to escape by the mere act of pursuit, whilst the other population, by this act of escape, provokes the chasers to exert themselves still further. It seems reasonable to assume (and many classical models have been based upon such assumptions) that the U's escape from the V's at a rate proportional to the spatial gradient of the V's, whilst the V's attempt to approach the U's at a rate proportional to the spatial gradient of the U's. Therefore, with α, β, γ, and δ constant, the population densities evolve according to

$$U_t + (\alpha - \beta V_x)U_x = 0,$$

$$V_t + (\gamma + \delta U_x)V_x = 0$$

Equations of this type were studied by Hsiao and DeMottoni [19]. Rescaling x and t and performing the transformation $U \to U/\delta$, $V \to V/\beta$, setting $U_x = u$ and $V_x = v$ we find that (with $a = \gamma/\alpha$)

$$u_t + [u(1 - v)]_x = 0, \tag{32}$$

$$v_t + [v(a + u)]_x = 0 \tag{33}$$

For $a > 1$, this system of conservation laws is elliptic when

$$D = (v - u + a - 1)^2 + 4u(a - 1)$$

is less than zero, and strictly hyperbolic when D exceeds zero.

3 9 *Relativistic radiative transfer*

Our final example emphasizes the fact that mixed systems of conservation laws may arise from completely unexpected sources. A study of general relativistic, frequency-dependent radiative transfer in spherical, differentially moving media was carried out by Turolla et al. [53] in which the moment equations were written as a system of conservation laws. Space does not permit an explanation of all the intricacies of the model, but the introduction of PSTF (projected symmetric trace-free) tensors allows two independent partial differential equations to be written for the first two moments w_0 and w_1 of the radiation intensity Physically, w_0 represents the radiation energy density, whilst w_1 denotes the radiation flux measured by a co-moving observer. In terms of the independent variables $t = \log r$ (the radial distance r may be thought of as being timelike in this treatment) and $x = \log \nu$ where ν denotes frequency, the equations may be written

$$(U^2 - 1)w_{0t} + \left(F - \frac{y'}{y}(U^2 - 1)\right)w_{0x} - \frac{c}{v}\,Gw_{1x} + C_1 = 0, \tag{34}$$

$$(U^2 - 1)w_{1t} - \frac{v}{c}\,U^2Fw_{0x} + \left(G - \frac{y'}{y}(U^2 - 1)\right)w_{1x} + C_2 = 0 \tag{35}$$

Here v denotes velocity, c is light speed, $U^2 = v^2(f_E + 1/3)/c^2$, $y = \gamma\sqrt{1 - r_g/r}$ (the total energy per unit mass), a prime denotes a derivative in the r direction, r_g is the gravitational radius, $\gamma = \sqrt{1 - v^2/c^2}$, and F, G, C_1, and C_2 are functions of the "Eddington factors" f_E and g_E which may be regarded as known functions and satisfy $0 \leq f_{\mathrm{E}} \leq 2/3$, $0 \leq g_{\mathrm{E}} \leq 2/5$. A standard characteristic analysis of (34) and (35) reveals that the relevant quadratic has discriminant

$$D = \frac{U^2}{(U^2-1)^2}\,(U^2(F^2-G^2)+4FG),$$

and although for many choices of the Eddington factors the system is strictly hyperbolic, it is possible that D may become negative. In particular, when the gravitational field is strong enough and/or the gas flow is almost in free-fall, the equations can become elliptic even for fairly modest values of v/c

4 Discussion and a qualitative comparison of the models described

It has been shown above how mixed systems occur in a variety of industrial and applied mathematical problems. It is natural therefore to ask what effect the appearance of complex characteristics is likely to have upon the analytical and numerical solutions of such conservation laws. From the simplest possible standpoint, we may anticipate that systems which possess complex characteristics will require different boundary conditions to hyperbolic problems. Presumably mixed problems require a specification of boundary and initial data which is somehow "in between" hyperbolic and elliptic problems. However, to quantify the proportions of Cauchy data and elliptic-type boundary data which are required for a given mixed problem is clearly a harder problem. We may also anticipate ill-posedness, since for example a Dirichelet problem for an elliptic system in time and space would necessarily involve the specification of both initial conditions and boundary conditions at later times, thus violating causality. Numerical problems may also be expected and it seems highly likely that even simple schemes that have proved reliable for hyperbolic equations such as the Lax–Friedrichs or MacCormack method (see, for example, [10]) may not be appropriate or even convergent for mixed problems. The same may well be true for more sophisticated schemes such as TVD methods [48] but it is likely that here the analysis will be even more complicated.

All of these problems arise when one of the independent variables of a mixed system is *timelike*. It is therefore the case that, although the transonic flow problem was included for the sake of completeness, it occupies a completely different position in the spectrum of mixed problems to the other models described above. Indeed, for the transonic flow problem, it is clear that the problem has an inherently mixed nature and any model that did not reflect this would have to be instantly rejected. Moreover, for this problem the remarks concerning ill-posedness simply do not apply; there is no reason why boundary conditions should not be specified on boundaries other than $x = 0$ say, and no question of causality being violated. Above all, we may be satisfied on the grounds of common experience that the appearance of both hyperbolic and elliptic regions in the flow is "physical". The same cannot be said for many of the other problems discussed in the previous section. Naturally, if we desire to solve the transonic flow problem with accuracy the mixed nature of the problem requires careful handling, for not only must it be ensured that only the physically correct shocks are selected, but also any numerical method that is employed must be able to cope with the change of type of the governing equations. The fact that entropy criteria, when suitably generalized, along with the existence of viscous profiles may still be used to select the "physically relevant" solution to the transonic flow equations was demonstrated by Mock [28] who was also able to prove a number of results concerning uniqueness. Numerically, many different methods [11, 29] have been successfully used to analyse transonic flows.

The traffic flow model is another matter altogether. Bick and Newell [3] were evidently surprised at the appearance of this non-trivial elliptic region of phase space, and commented, "As yet we have found no satisfactory explanation of why the equations should be elliptic nor any satisfactory explanation as to what one should do about it." An appealing explanation of the non-hyperbolicity is that the constitutive laws (14) are somehow at fault, but it is a source of some worry that Bick and Newell were able to show by elementary means that the region of ellipticity persists under much more general constitutive laws. They did not pursue the point, however, preferring to consider properties of the solutions of the equations only for the cases where the phase paths avoided the elliptic region. One final comment is apposite; the model that was employed treated the discrete entities of cars as a continuum, so that in some sense "averaging" was employed. It will become apparent that the use of averaging frequently presages the occurrence of mixed type conservation law systems.

The equations for flow of a van der Waals gas and those for elastic waves obviously pose closely related problems. One fact is clear for both problems, however; if only the standard Rankine–Hugoniot and entropy conditions are used, then non-uniqueness will result. As a result of this, much work has been carried out to try to formulate admissibility criteria that are capable of selecting the "physically correct" solution. One criterion, based upon physical grounds, was suggested by Hattori [14], who insisted that, for a solution to be admissible, the entropy should decrease at the greatest rate possible. Using this, he was able to show how to construct solutions to the mixed problem. A further study [15] showed that the entropy rate admissibility criterion was also relevant for the nonisothermal van der Waals problem for solutions that consisted only of a backward and a forward wave, a contact discontinuity and a phase boundary. As this by no means covers all solutions to this problem, however, the entropy rate admissibility criterion cannot be regarded as a complete answer to the admissibility problem.

Returning to the standard van der Waals/elasticity problem, many other studies have concerned themselves with the issue of admissibility. Shearer [42] used the viscosity admissibility criterion, but was able to exhibit Riemann problems that possessed two admissible solutions.

The inclusion of both viscous and capillarity effects in the formulation of an admissibility criterion was considered by Slemrod [44], who analysed solutions to the equations

$$w_t - u_x = 0,$$

$$u_t + p(w)_x = \varepsilon u_{xx} - \varepsilon^2 A w_{xxx} - 2\varepsilon^2 D w_x w_{xx}$$

where A and D were constants. Arguing that the standard viscosity admissibility criterion was too restrictive in that it did not allow propagating phase boundaries, he showed that, under certain special circumstances, such boundaries were permitted if the viscosity/capillarity condition was employed.

Another approach to the elasticity/van der Waals flow problem has been to argue that it is physically incorrect to *specify* the pressure as a function of the strain. Instead, it should satisfy a conservation law. This approach was pioneered in [51] and a specific case of the elasticity problem was examined by Pitman and Ni [32]. They proposed that the standard equations for the velocity v, strain u and pressure p

$$u_t - v_x = 0, \quad v_t - p_x = 0$$

should be closed not by assuming that p was given by some reference pressure $p_r(u)$ (the classic case being $p_r(u) = u^3/3 - u$, elliptic for $-1 < u < 1$), but by a conservation law of the type

$$(p - E^2 u)_t = -(p - p_r)/\tau$$

Here E is the Young's modulus of the material and τ is a relaxation time. This changes the eigenvalues of the system to 0 and $\pm E$, a convenient result which was exploited to allow stable numerical results to be computed. As we shall see, the approach of "adding some extra equations" to ensure hyperbolicity has frequently been employed (with varying degrees of success) in the two-phase flow problem.

The number of different methods that have been used to analyse the van der Waals flow/elasticity problem makes one fact quite clear; there is a general lack of agreement concerning the right way to approach this mixed system.

The mixed nature of the oil recovery problem has had profound implications, for numerical solutions to such equations are much sought-after by industry. For this reason, many practical calculations have simply proceeded using ad hoc methods, obtaining numerical results by any means possible. Another alternative is to modify the experimental correlations that determine the relative permeabilities to ensure that no elliptic regions are present. Under the assumption that the relative permeabilities were specified in such a way that the system is hyperbolic, Trangenstein and Bell [52] showed that fronts may be computed successfully. Another alternative to changing the correlations is to change the model, and a study by Stevenson et al. [49] which addressed general methods for the problem suggested that capillary or diffusive pressure terms should be included in the model to ensure well-posedness. An investigation into the essential nature of the mixed problem was carried out by Keyfitz [23] who proposed a 2×2 analogue problem that possessed many of the features of the oil recovery problem. She concluded that in some cases the Riemann problem could be solved.

Time is not one of the independent variables in the shear-thinning flow described by Eqs. (24)–(26), and so, as in the transonic flow case, the normal comments concerning ill-posedness and violation of causality do not apply. The example has been included, however, to indicate that, even when a change of type occurs in what might be termed as benign circumstances, it can nevertheless cause major practical difficulties. In this case, to analyse the zeroth-order approximation to the flow for small ε, we are forced to examine a system whose character is different to that which is retrieved for any $\varepsilon > 0$ In a sense, the normal problems of mixed systems have been reversed in that the "full" system is elliptic, but the limit is hyperbolic. Real characteristics must therefore somehow be "embedded" in an elliptic domain. Work in progress on this problem suggests that, though this can be accomplished, the details are complicated.

The two-phase flow equations (27) to (30) have been used as a model for many different flows, ranging from stratified flow (where there is only one phase boundary) to bubbly and gas/particulate flows. Since it is unreasonable that the same set of equations should describe such patently different flows, it may be regarded as hardly surprising that the model possesses some awkward properties. Two-phase flow models will be treated in detail in subsequent sections, and for the moment we observe only that, since elementary hydrodynamics asserts that the pressure on the surface of a sphere in a uniform flow differs from the pressure at infinity, the equal pressure hypothesis that has been used to close the model (27)–(30) is clearly wrong.

The simplicity of the model that leads to the population dynamics equations (32) and (33) shows how easily and naturally mixed systems may arise. The familiar mixed system properties of a simplified constitutive assumption and a model that is averaged in some sense arise once again, and, as usual, the goal is to make some sense of Riemann problems for the system. It was shown in [19] that is possible to combine the Lax and Liu–Oleinik entropy conditions to propose a GEC (generalized entropy condition) which is enough to guarantee existence (but not uniqueness) of the solution to the Riemann problem. When a criterion was added stating that the sum of the strengths of all jumps in the solution should take the minimum value amongst all possible jumps belonging to a certain set, uniqueness could also be demonstrated.

As far as the relativistic radiative transfer case is concerned, the authors do not comment at length upon the appearance of a mixed system, save to point out that the presence of complex characteristics may have a profound influence on the boundary conditions that have to be imposed. Bearing in mind the fact that the system could be of mixed type for a range of apparently reasonable physical assumptions, and also that the model represents a simple attempt to describe a hugely complicated range of phenomena, it seems that, as in the case of two-phase flow, the problem can be best categorized as arising due to *modelling* concerns. It should also be noted that only two of the (theoretically infinite in number) moment equations were studied. Evidently if more were used and these contained submodels that introduced back-coupling terms, yet further changes in type could occur in the equations.

The role of viscosity

Before discussing the work that has been carried out on prototype mixed systems, it is pertinent to discuss the role of viscosity in mixed systems A line of reasoning that has frequently been proposed to circumvent all the difficulties of mixed systems has been simply to argue that, in reality, physical systems actually involve dissipative mechanisms such as viscosity. The introduction of higher derivative terms into the models is therefore necessary if they are to be physically realistic, and for this reason any time spent studying inviscid models and all the attendant complications of mixed systems is wasted.

There are at least three good reasons for rejecting this line of argument, which may be summarized as follows:

1. Mixed systems arise in such a range of applications that it may not be clear just *what* the correct dissipative mechanism actually is. As an example, in the population dynamics model discussed above it is not at all clear what physical mechanism would play the role of viscosity.

2. It is common experience that, for single-phase fluid flows, viscosity may essentially be ignored except in the boundary layer close to a solid boundary in the flow. The inviscid theory of shock waves has proved eminently capable of accurately predicting the properties of viscous flows everywhere except for the thinnest of regions, where viscosity is important. If the single-phase inviscid theory is so successful, then why should inviscid theory for multi-phase systems be so fatally flawed, and why should we be forced to include small viscous terms to have any hope of obtaining a well-posed problem?

3. Consider the system of equations

$$u_t + p'(v_0)v_x = \varepsilon u_{xx}$$
$$v_t - u_x = \delta v_{xx}$$

where the nonlinear term $p(v)_x$ has been "frozen" at v_0 and ε and δ represent small dissipative terms. We examine the effect of introducing a Fourier mode $(u,v)^T = (u_0, v_0)^T \exp(\omega t + ikx)$ into the solution. A simple calculation shows that for non-trivial solutions we require that

$$(\omega + \varepsilon k^2)(\omega + \delta k^2) - k^2 p' = 0,$$

and thus

$$\omega = \frac{-k^2(\varepsilon + \delta) \pm \sqrt{k^4(\varepsilon+\delta)^2 - 4(\varepsilon\delta k^4 - p'k^2)}}{2}.$$

Therefore $\mathrm{Re}(\omega) > 0$ whenever $k^2(\varepsilon + \delta) < 0$ and/or $p' > \varepsilon\delta k^2$. The first of these conditions simply confirms that negative viscosity leads to an ill-posed problem, but the latter condition shows that, if ever $p'(v_0) > 0$ (and thus the inviscid system is mixed) then there are always unstable wavenumbers. Many other similar calculations may be carried out including extra higher order derivative dispersive terms (representing, for example, capillary effects) but the general conclusion is that the complex eigenvalues of the inviscid system may "pollute" associated viscous systems, a fact that may easily be confirmed via numerical calculations.

5 Prototype mixed conservation laws

The mixed systems discussed above are all complicated and in most cases it is extremely unlikely that useful closed-form solutions can be determined. In order to understand some of the complicated behaviour that such systems may exhibit, it is obviously an attractive idea to consider paradigm mixed problems This approach has been followed by a number of authors, in many cases with some unexpected results. Some illustrative mixed prototype problems will be discussed below, and some attention will also be given to the related problem of nonstrictly hyperbolic mixed systems.

5 1 Mixed elliptic/hyperbolic prototype systems

Prototype equations that have been studied have consisted almost invariably of 2×2 systems whose flux terms are chosen to give rise to specific mixed regions in phase space. To allow shocks to be analysed for mixed systems, some further generalizations of (8) and (9) are required. In most (but not all) examples, the Rankine–Hugoniot condition is used unchanged in the form (8), but clearly for complex eigenvalues the entropy condition can no longer be used in the form (10).

The most common extension of the entropy condition has become known as the Liu–Oleinik criterion [26]. This asserts that, for all $\mathbf{w}$ on the curve in phase space joining $\mathbf{w}_L$ to $\mathbf{w}_R$ (the Hugoniot curve), it must be true that

$$s(\mathbf{w}_L, \mathbf{w}_R) \leq s(\mathbf{w}_L, \mathbf{w}).$$

Although this is an obvious extension of the Lax condition, we note that it relies upon the Hugoniot curves being both connected and non-intersecting.

The Liu–Oleinik entropy criterion was used to study [16] the Riemann problem for the prototype mixed system

$$\mathbf{w}_t + \mathbf{f}(\mathbf{w})_x = 0$$

where $\mathbf{w} = (u, v)$ and $\mathbf{f} = (v^2 - u^2 + 2\rho v, 2uv - 2\rho u)$. The specific construction of this system ensures that the equations are elliptic inside the region E_ρ where $u^2 + v^2 < \rho^2$ and hyperbolic when $u^2 + v^2 \geq \rho^2$.

Even for the relatively simple elliptic region defined by this problem, it transpires that the combination of shocks and rarefaction waves required to effect a complete solution of the problem is rather involved. Nevertheless, it can be constructed. Although (as might be anticipated) solutions are not necessarily continuous in (u_L, v_L) and (u_R, v_R), all approach the "correct" solution as $\rho \to 0$, and in many of the different regions that must be considered the solution is essentially unique. However, when the left state is close to E_ρ a connection may be made to the right state in two ways, for as well as the classical Lax solution another solution may be constructed using five shocks to traverse around E_ρ. Although it might be possible to establish uniqueness via viscous profile arguments, the calculations required are evidently formidable.

A related prototype Riemann system was studied by Holden and Holden [17] In this case, the region of ellipticity in phase space was the interior of the ellipse $u^2 + 16v^2 = 16r^2$. Once again, the generalization and effective weakening of the entropy condition that is required to assure the existence of a solution leads to non-uniqueness in some regions; other regions were also found where there was no solution. The solution was further complicated by the need to include multiple composite waves. As an example, in some regions rarefaction/shock/rarefaction (RSR) and rarefaction/shock/rarefaction/shock (RSRS) structures are both necessary to obtain a solution, emphasizing the contrast with the maximum of two waves required for the strictly hyperbolic 2×2 Riemann problem.

A more general study of such problems was carried out by Holden et al. [18], who studied qualitative features of the solution of the Riemann problem for a generic mixed system with both hyperbolic and elliptic regions. They were able to show that if the left and right states were in the elliptic region and the elliptic region was connected, then the solution necessarily contained parts which lay outside the elliptic region. In contrast, if both states were outside the elliptic region the solution did not enter the elliptic region. Although this is an interesting and potentially valuable result, they were also able to show that the latter property did not hold for general Cauchy data, thus further complicating the issue.

Other mixed elliptic/hyperbolic systems have also been examined: the study by Fitt [7] gave the solution to a prototype p-system which was hyperbolic in the lower half-plane and elliptic in the upper half plane. Shearer [40] analysed yet another mixed p-system where the eigenvalues were complex in a strip in phase space, defining shocks to be admissible if either the Oleinik entropy condition was satisfied, or if the shock speed was zero. Another very complicated structure emerges, some 18 regions being necessary. In this case, the price that had to be paid to display a complete solution to the problem was the inclusion of stationary shocks; no viscous profile could be constructed for these, however, and this gives rise to serious worries about the physical basis of such discontinuities.

Taking account of these and the many other mixed systems that have been analysed, no clear picture emerges. It is also evident that, though many different approaches have been proposed for admissibility criteria, no single condition has universal applicability. In view of the relative lack of progress for 2×2 systems, it is likely that the solution structure of the elliptic/hyperbolic equations for two-phase flow will remain unknown for the foreseeable future.

5.2 Mixed systems that are not elliptic

As discussed above, the main focus of this survey concerns systems of conservation laws that are elliptic in some regions of phase space. Nevertheless, there is also considerable interest in nonstrictly hyperbolic and mixed hyperbolic/parabolic systems where the eigenvalues never become complex but are equal in some regions of phase space. One of the great values of studying such systems is that they may suggest new shock structures which may themselves be valuable for mixed hyperbolic/elliptic cases.

Keyfitz and Kranzer [25] solved the Riemann problem for a class of hyperbolic conservation laws that possessed a parabolic degeneracy in the form of a curve in the (u, v)-plane where the two eigenvalues were equal and possessed a single common eigenvector. Physically, the most interesting such genuinely nonlinear prototype system is defined by

$$u_t = v_x, \quad v_t = (u^3/3)_x,$$

which is tantamount to a nonlinear wave equation $u_t = (c^2 u_x)_x$ with the sound speed $c = u$. The degeneracy occurs when the sound speed is zero. To solve the problem completely, phase space must be divided into twelve regions. In six of these regions two waves are required, as in the strictly hyperbolic case. However, in five of the remaining regions the solution necessarily involves three waves, whilst in the twelfth and final region four waves are required. The "extra" waves invariably involve rarefactions that begin or end in shocks. Another nonstrictly hyperbolic class of 2×2 systems arising from nonlinear wave propagation on an elastic string was studied by Keyfitz and Kranzer [24]. Once again, there were regions of phase space where three rather than two waves were required for a solution, though in this case the weak nonlinearity of the equations guaranteed that one of the waves was always a contact discontinuity.

In view of the continuing interest in nonstrictly hyperbolic systems, some attempts at a general classification of phenomena have been made. A general treatment of nonstrictly hyperbolic conservation laws with quadratic flux functions was carried out by Isaacson and Temple [21]. It is clear from the involved nature of the calculations required, however, that a complete classification will be a major undertaking. A particular case with quadratic flux functions was examined by Schaeffer and Shearer [37], who studied the Riemann problem for a system with flux $\mathbf{f} = (au^2 + 2buv + v^2, bu^2 + 2uv)$. The eigenvalues are real, but equal at the origin. (An "isolated umbilic point") In all there are four cases to consider depending on the constants a and b. In contrast to the strictly hyperbolic 2×2 Riemann problem, where for a fixed left state the right state plane divides into four regions, here there are fifteen regions to consider. In each region the solution is composed of combinations of shocks, rarefactions and composite waves where shocks and rarefactions occur together. Furthermore, *overcompressive* shocks (both characteristics on both sides of the shock enter the shock) must be introduced for some types of data.

When the flux functions are not quadratic, more exotic solution structures may be present, as shown by the work of Schaeffer et al. [39]. They studied the system

$$u_t + (u^2 - v)_x = 0, \tag{36}$$

$$v_t + \left(\frac{\gamma}{3} u^3\right)_x = 0, \tag{37}$$

which has eigenvalues

$$\lambda_\pm = u \pm |u|\sqrt{1-\gamma}.$$

Thus for $\gamma < 1$ the eigenvalues are equal only when $u = 0$, so that strict hyperbolicity fails on a line in phase space. Using the regularization obtained by adding terms εu_{xx} and εv_{xx} respectively to (36) and (37), it was possible to show that whilst for $\gamma < 0$ the Riemann problem could be solved in terms of elementary waves, for $0 < \gamma < 1$ it was necessary to introduce "singular" shocks. Such shocks satisfied generalized Rankine–Hugoniot conditions of the form

$$-s(u_L - u_R) + u_L^2 - u_R^2 - (v_L - v_R) = 0$$

$$-s(v_L - v_R) + \tfrac{1}{3}\gamma(u_L^3 - u_R^3) > 0$$

The fact that the second condition appears in the form of an inequality (rather than the usual equality) proved to be enough to specify the inner solution uniquely. One interpretation of this result is that it confirms that there are many different ways of slightly altering the admissibility conditions of the solution; what is less clear is the physical basis for these changes and how they are all related.

The literature on nonstrictly hyperbolic systems is very large, and for additional information, some of it related to the oil recovery problem, the reader is referred to [43, 38, 41]. The main message seems to be that an ad hoc approach is usually necessary, and systems that are superficially almost identical may nevertheless possess very different shock structures.

6 Mixed systems in two-phase flow

Although many examples have been discussed above, we wish now to discuss in particular the specific case of two-phase flow Such flows arise with great frequency in industrial processes, and hence a huge amount of relevant literature exists Nevertheless, only a small proportion of published studies acknowledge the problem of non-hyperbolicity

As far as practically important two-phase flow systems are concerned, it is not too great a simplification to divide the most frequently employed methods of solution of such problems into three classes. Firstly, it is possible to add terms piecemeal to a mixed two-phase flow system, until the system has the properties that are desired. Secondly, a radical approach may be taken and the modelling may be re-examined from its most basic assumptions. Finally, the problem of non-hyperbolicity may be completely ignored, and numerical methods (which, of necessity, are supplemented with large amounts of artificial viscosity or other "smoothing" strategies) may be used.

6.1 "Modifications" to standard models

There are a number of ways in which the equations of two-phase flow may be "modified" in order to circumvent the problem of complex characteristics. At the simplest possible level, it has frequently been noted that if the terms $\alpha_1 p_x$ and $\alpha_2 p_x$ in Eqs (29) and (30) are replaced respectively with $(\alpha_1 p)_x$ and $(\alpha_2 p)_x$, then the characteristics of the resulting system satisfy the quadratic equation

$$\lambda^2(\alpha_1\rho_2 + \alpha_2\rho_1) - 2\lambda(u_1\rho_1\alpha_2 + u_2\rho_2\alpha_1) + \alpha_1\rho_2 u_2^2\alpha_2\rho_1 u_1^2 - p = 0,$$

and are thus real if

$$p(\alpha_1\rho_1 + \alpha_2\rho_2) > \rho_1\rho_2\alpha_1\alpha_2(u_1 - u_2)^2$$

This implies that, provided the relative velocity is low enough, the model is strictly hyperbolic. Any potential advantages of this result are completely negated when it is noted that, quite apart from being based on physically dubious principles, the equations do not possess steady-state solutions where the void fraction is non-uniform in space. Significantly, when these "modified" equations were used in a study of military internal ballistics, they caused the weapon to numerically "fire" *before* the trigger was engaged.

The single-pressure hypothesis was identified fairly early on in the study of two-phase flow systems as the possible root cause of the appearance of complex characteristics. Because of this, many studies have been carried out where independent phasic pressures were assumed. Of course, if two pressures are used then the main problem becomes one of closure since the system is now an equation short. Often closure has been achieved by proposing a "void fraction propagation" equation. A typical model of this sort was studied (amongst others) by Ransom and Hicks [36]. They modelled compressible planar separated flow in a channel of width H using the equations

$$(\rho_1\alpha_1)_t + (\rho_1\alpha_1 u_1)_x = 0,$$

$$(\rho_2\alpha_2)_t + (\rho_2\alpha_2 u_2)_x = 0,$$

$$(\rho_1\alpha_1 u_1)_t + (\rho_1\alpha_1 u_1^2)_x + \alpha_1 p_{1x} + (p_1 - \tilde{p})\alpha_{1x} = 0,$$

$$(\rho_2\alpha_2 u_2)_t + (\rho_2\alpha_2 u_2^2)_x + \alpha_2 p_{2x} + (p_2 - \tilde{p})\alpha_{2x} = 0,$$

$$\alpha_{1t} + \tilde{u}\alpha_{1x} = \tilde{v}/H$$

where $\tilde{u} = (u_1 + u_2)/2$, $\tilde{v} = (p_1 - p_2)/(a_1 + a_2)$, $\tilde{p} = (p_1 a_2 + p_2 a_1)/(a_1 + a_2)$, and $a_n = c_n\rho_n$, the c_n being the respective speeds of sound. The eigenvalues of this set of equations are easily shown to be given by $\tilde{u}$ and $u_n \pm c_n$, and the problem of non-hyperbolicity does not arise. Although this system looks appealing, there seems to be no rational averaging procedure that gives the void fraction propagation equation. There is also some doubt about the reasoning behind the specific form of the pressure terms in the momentum equations. In addition to these worries, it may be shown [33] that no uniform steady flows are possible. We conclude that whilst theoretically and numerically the equations are easy to use, it is impossible to be confident that the equations provide a good model of the physical processes that are taking place.

There are many alternatives to the possibilities described above. Simply omitting the pressure term from the phase 2 momentum equation (30) leads to a system with characteristics 0, 0, u_2 and u_2. Because of this desirable property many arguments have been put forward to justify the neglect of this pressure term. Although there are undoubtedly circumstances where this procedure may be justified on sound dimensional grounds (for example, the flow of a "dusty" gas as studied in [27, 20]), there is no compelling physical reason to omit this term for other two-phase flows.

Another common approach is to assert that "some of the physics has been omitted". For example, in the case of two-phase gas-particulate flow, it might be argued that since "the gas is clearly not incompressible" an energy equation of the gas phase and some sort of gas constitutive law should be added. To examine the effect of this, let us suppose that the variable ρ_1 is regarded as non-constant, and the system (27)–(30) is augmented by the addition of the equation

$$(\rho_1\alpha_1E_1)_t+\left(\rho_1\alpha_1u_1\left(E_1+\frac{p}{\rho_1}\right)\right)_x+p(\alpha_2u_2)_x=0 \tag{38}$$

where $E_1=e_1+u_1^2/2$, e_1 denotes internal energy, and the system is closed by invoking the perfect gas law. Calculation of the characteristics λ is simplified in this case by setting $\lambda=yc+u_1$, where $c^2=\gamma p/\rho$ and γ is the specific heat ratio. After a standard calculation, we find that y satisfies

$$y(y^4-2Vy^3+y^2(V^2-q-1)+2Vy-V^2)=0 \tag{39}$$

where $V=(u_2-u_1)/c$ and $q=(\alpha_2\rho_1)/(\alpha_1\rho_2)$. Analysis of the quartic part of (39) shows that for non-zero V, the equation has four real roots if and only if

$$V^2>(1+q^{1/3})^3, \tag{40}$$

and if this condition does not hold then there are two real and two imaginary sound speeds. Although this may be interpreted as having "improved" the situation slightly, we note that in all examples where both phases are stationary, V will not satisfy (40) and the solution will inevitably pass through an elliptic region of phase space.

Other attempts to render the equations hyperbolic have used a variety of additional equations: Ramshaw and Trapp [35] added the effects of surface tension, Stuhmiller [50] included additional modelling arising from interfacial pressure terms, Prosperetti and Van Wijngaarden [34] proposed a model specifically tailored to the case of bubbly flow, whilst Drew et al. [5] considered the additional effects of virtual mass. Taken as a whole, these studies seem to suggest that the inclusion of no single extra effect can completely remedy the problem of non-hyperbolicity

The examples given above represent only a small fraction of the totality of attempts that have been made to solve the problem of complex eigenvalues by augmenting or changing the equations of motion in a piecemeal fashion. Although there are obviously circumstances in which this may be a useful approach, it is evident that, to provide a long-term answer to the problem, deeper study is required. The example given immediately above points the way forward; the key step consists in realising that the averaging process that is inherent in the formulation of any two-phase flow equations has led to terms being "left out" of the equations. Any attempt to correct the situation will therefore require a consideration of the averaging process from its first stages.

6.2 *A radical modelling approach*

The first successful attempt to rationalize the averaging process and propose a full set of equations of motion for two-phase flow was that of Drew and Wood [6]. They realized that, rather than adding equations to the system as and when they were needed, the modelling needed to be considered afresh. With this in mind, they returned to the original flow conservation laws in the form

$$(\rho\Sigma)_t = \nabla\cdot(\rho\mathbf{q}\Sigma - \mathbf{J}) = \rho f \tag{41}$$

Here an arbitrary conserved quantity is denoted by Σ, its velocity by $\mathbf{q}$, its density and molecular flux by ρ and $\mathbf{J}$ respectively, whilst f is the relevant source density (if any). Merely satisfying (41) for each conserved quantity in each phase is not sufficient to close the model, since the boundaries between the phases are free. The model is therefore completed by specifying interfacial jump conditions in the form

$$f_i = [(\rho\Sigma(\mathbf{q} - \mathbf{q}_i) + \mathbf{J})\cdot\hat{\mathbf{n}}_i]_1^2 \tag{42}$$

where the subscript i denotes an interfacial quantity and $\hat{\mathbf{n}}$ is the unit normal.

To transform (41) and (42) into one-dimensional conservation laws, an involved process now ensues. The equations are successively ensemble averaged and cross-sectional area averaged, and the jump conditions are used to ensure that interfacial quantities are suitably treated. Omitting all the details (see [6, 9]), the end result for one-dimensional flow in a channel of slowly-varying width is the equations

$$(\alpha_k\rho_k)_t + \frac{1}{A}(A\alpha_k\rho_k u_k)_x = \Gamma_k + f_{mk}, \tag{43}$$

$$(\alpha_k\rho_k u_k)_t + \frac{1}{A}(A\alpha_k\rho_k u_k^2 C_{uk})_x = \frac{1}{A}(A\alpha_k(T_k + T_k^{Re}))_x + M_k$$
$$+ \frac{4}{D}\alpha_{kw}T_{kw} + u_{ki}\Gamma_k + \alpha_k\rho_k f_{pk}, \tag{44}$$

$$(\alpha_k\rho_k(e_k + e_k^{Re} + u_k^2/2))_t + \frac{1}{A}(A\alpha_k\rho_k u_k C_{ek}(e_k + e_k^{Re} + u_k^2/2))_x$$
$$= -\frac{\xi_h}{A}\alpha_{kw}\zeta_{kw} + \frac{1}{A}(A\alpha_k((T_k + T_k^{Re})u_k - \zeta_k - \zeta_k^{Re}))_x + G_k + W_k$$
$$+ \alpha_k\rho_k(r_k + u_k f_{pk}) + (e_{ki} + u_{ki}^2/2)\Gamma_k \tag{45}$$

In this formidable set of equations, k takes the value of 1 or 2 for the continuous and dispersed phases respectively, the subscript w indicates wall properties and the superscript Re denotes turbulent quantities. As far as the variables are concerned, A is the channel cross-sectional area and D the effective channel diameter, T represents stress, Γ is the interfacial heat source, f_m and f_p are respectively the bulk mass and momentum sources, and M, G, and W represent respectively interfacial force, heat source, and work. The internal energy is denoted by e, source heating by r, channel perimeter heating by ξ, and axial energy flux by ζ. Finally, the profile coefficients C_u and C_e account for differences between the average of products and the products of averages.

Equations (43)–(45) provide a model for *all* two-phase flows; to close the model, however, a number of constitutive assumptions must be made. Evidently these submodels will be different for different types of flow A rational way of approaching the modelling is to non-dimensionalize, compare terms, and proceed via traditional asymptotics. The hope then is that *if* the right dominant physical effects can be identified and *if* accurate submodels can be proposed for these effects, a strictly hyperbolic system should result.

Although this is an appealing programme of work, the complications involved should not be underestimated. Drew and Wood [6] remark that, "There are 62 variables and parameters described by as many relations: 6 conservation equations, 3 jump conditions, 4 equations of state, 2 equations of phase contiguity and 47 closure conditions. The formulation is nominally complete." Most importantly, the emphasis has shifted to a need for accurate submodelling, and if models are proposed that are insufficiently realistic, then it is hardly surprising if the result is a mixed system.

Although some progress has been made in examining the effects of the added terms in the equations (see, e.g., [9]), much work remains to be carried out. To give an example of the sort of result that may be obtained, consider a laminar two-phase bubbly flow where both phases are incompressible and the effects of interfacial pressure differences are included. The relevant equations are

$$(\rho_1\alpha_1)_t + (\rho_1\alpha_1 u_1)_x = 0,$$

$$(\rho_2\alpha_2)_t + (\rho_2\alpha_2 u_2)_x = 0,$$

$$(\rho_1\alpha_1 u_1)_t + (\rho_1\alpha_1 u_1^2)_x + \alpha_1 p_{1x} = -C_s\rho_1(u_1 - u_2)^2\alpha_{1x},$$

$$(\rho_2\alpha_2 u_2)_t + (\rho_2\alpha_2 u_2^2)_x + \alpha_2 p_{1x} = \alpha_2 C_s\rho_1(u_1 - u_2)^2_x,$$

the assumption having been made that the bulk and interfacial pressures are related according to

$$p_2 = p_{i2} = p_{i1} = p_1 - C_s\rho_1(u_1 - u_2)^2$$

where $C_s < 0$ is a pressure coefficient that depends on the shape of the bubbles.

Carrying out a hyperbolicity analysis as usual, we find that $\lambda = 0$ (twice) and

$$\begin{aligned}(-\alpha_1\rho_2 - \alpha_2\rho_1)\lambda^2 &+ 2(C_s\rho_1(u_1 - u_2) + u_2\rho_2\alpha_1 + u_1\rho_1\alpha_2)\lambda \\ &+ C_s\rho_1(u_1 - u_2)(-\alpha_2 u_1 - u_2\alpha_1 - u_2) - \alpha_1\rho_2 u_2^2 - \alpha_2\rho_1 u_1^2 = 0. \qquad (46)\end{aligned}$$

The discriminant D of (46) is given by

$$D(\alpha_1) = -4\rho_1(u_1 - u_2)^2[-\rho_1 C_s^2 + (-\alpha_1^2\rho_2 + \rho_1\alpha_1^2 + \alpha_1\rho_2 - \rho_1)C_s + \alpha_1\alpha_2\rho_2],$$

and the equations are thus hyperbolic if the term in square brackets above is less than zero. To analyse this case, assume that $\rho_2 = R\rho_1$ where $R \ll 1$ (for gas bubbles in a fluid typically $R \sim 10^{-3}$). Then since $D(0) = -C_s\rho_1$ and $D(1) = -\rho_1 C_s^2$, hyperbolicity will be determined by the solutions of the equation

$$(-C_s + R(1 + C_s))\alpha_1^2 + R(-1 - C_s)\alpha_1 + C_s^2 + C_s = 0.$$

For small R, the obvious perturbation solution gives

$$\alpha_1 = \sqrt{1 + C_s} + R\,\frac{(1 + C_s)(-1 + \sqrt{1 + C_s})}{2C_s} + O(R^2)$$

Inclusion of the interfacial pressure coefficient thus ensures hyperbolicity whenever α_1 lies between the above value and unity. For example, with $R = 1/1000$ and $C_s = -7/16$ (spherical bubbles) we find that the equations are hyperbolic whenever $0.75 < \alpha_1 \leq 1$.

This result has a number of appealing properties. First, although only one extra term has been included, the hyperbolicity region in phase space has been increased in size. Moreover, because of the careful modelling involved, it is possible to specify *exactly* the sorts of flows for which this result is relevant, in that the interfacial pressure non-dimensional group exactly balances the convective terms, and exceeds all others in order of magnitude. Finally, the hyperbolicity result suggests a physically correct conclusion, namely, that when the volume fraction of bubbles becomes larger than around 1/4, another flow regime is entered and other terms (for example bubble/bubble collision exchange terms) must be included.

6.3 Pragmatic numerical approaches

In the previous discussion a methodology has been outlined that must be regarded as the correct way forward. Proceeding on purely pragmatic lines, however, it is clear that for many industrially-important mixed problems the complexity of the flows involved may mean that rigorous mathematical modelling will prove to be either impossible or prohibitively time-consuming. Because of this, standard numerical methods have frequently been used to solve mixed problems, and all the difficulties associated with violation of causality, ill-posedness, and instability have been ignored.

Some attempts have been made to examine the numerical properties of mixed systems. Pego and Serre [31] used Glimm's scheme for the mixed elasticity problem, concluding unsurprisingly that strong convergence of the scheme could not be obtained. They also carried out numerical experiments, observing macroscopically random transitions to nearby states and other types of non-convergence. Affouf and Caflisch [1] considered the viscosity/capillarity regularization of the van der Waals flow mixed problem in the form

$$v_t - u_x = 0\,, \quad u_t + p(v)_x = \varepsilon u_{xx} - \delta v_{xxx}$$

with $p(v) = v - (v-2)^3$, and were able to successfully compute numerical solutions for some values of δ and ε. In general, however, the problem has been recognized as a formidable one.

If we accept that the need to solve real problems makes it inevitable that "dirty" numerical methods will be used, knowing the extent to which the numerical results that are produced may be trusted becomes a priority. Although the results may possess small errors, can we be sure that the predicted general trends are correct? In general, we must conclude that the answer is no. In spite of this, a wide range of numerical studies have been carried out on mixed systems of two-phase flow equations of varying degrees of complexity. Whilst it is certainly true that many industries (two specific examples are the nuclear power industry and the defence industry) solve ill-posed systems of equations daily and base strategic decisions upon the results, there are also many cases where incorrect predictions are made. Another problem that may prove just as serious stems from the fact that, in order to "encourage" numerical methods for mixed problems to converge, large amounts of numerical dissipation ("artificial viscosity") are often employed. In the worst cases, this leads to shock waves that are so smeared as to be almost unidentifiable.

Even bearing in mind the caveats of the above discussion, numerical methods are easy to carry out and show that, under some circumstances, sensible results may be obtained. For example, the Lax–Friedrichs method

$$\mathbf{w}_k^{n+1} = \frac{1}{2}(\mathbf{w}_{k+1}^n + \mathbf{w}_{k-1}^n) - \frac{\mathrm{d}t}{2\mathrm{d}x}(\mathbf{f}_{k+1}^n - \mathbf{f}_{k-1}^n) \tag{47}$$

for

$$\mathbf{w}_t + \mathbf{f}(\mathbf{w})_x = 0$$

was used in [7] to solve a mixed p-system where the solution was known. For the method described by (47), $x = k\mathrm{d}x$, $t = n\mathrm{d}t$ and the Courant number $\mathscr{C} = \max|\lambda_k|\,\mathrm{d}t/\mathrm{d}x$ was fixed. The choice of this particularly simple first order method was deliberate, and was made so that no artificially introduced dissipative mechanisms would pollute the results. Because of the low order nature of the scheme, a large number (typically thousands) of spatial mesh points were used. The results showed that, when both states lay in the hyperbolic region, the numerical solution was reliable and accurate. For some cases where one state lay in the elliptic region and the other in the hyperbolic region, the solution was acceptable, but displayed unexpected oscillations. (The numerical scheme is monotone, and for strictly hyperbolic systems is thus guaranteed oscillation-free.) The general conclusion of this study was that, for the majority of left and right states the numerical solution, though somewhat flawed, described the "known" solution with tolerable accuracy.

Numerical studies were carried out on the two-phase flow system (27)–(30) and (38) in [7, 8]. Numerically, this is a particularly convenient system to study, since, although there are five dependent variables, the ellipticity condition is described by only two parameters, leading to an effectively two-dimensional phase space. The most striking result of this investigation was that, when left and right states were chosen close together in the elliptic region, the solution developed large oscillations (similar to those noted in the three-phase oil recovery problem [2]) which wandered ever further away from the initial states in phase space. The results looked tolerable for some connections between hyperbolic and elliptic phase space, however. We conclude that numerical calculations, whilst difficult, may not be impossible to perform.

7 Mixed problems: the future

Ample evidence has been presented that mixed systems of conservation laws present serious problems in many different industrially-related mathematical models. They do not occur only in isolated and atypical circumstances, and indeed may appear from the simplest and ostensibly well-motivated of models. It is therefore relevant to consider the future of the subject, discuss what has been achieved, and indicate what work remains to be carried out.

As far as theory is concerned, it is certain that much more remains to be achieved. The many studies of prototype systems described above have served to emphasize that, when an evolutionary system has complex (or even real, but coincident) eigenvalues, the correct conditions that are required to ensure existence and uniqueness are not known. The fact that so many studies of prototype systems have used such a wide variety of replacements for the Rankine–Hugoniot and entropy conditions confirms that there is no real agreement or general theory for mixed system admissibility criteria, and until such a theory exists the study of prototype problems will remain in its infancy. Unfortunately, most of the existing evidence suggests that general results are far away at the present time. Mathematically, the exotic shock structures that form part of mixed system solutions are highly interesting. There seems to be no suggestion, however, that they have ever been observed experimentally, and if this could be accomplished then it would represent an important step.

In the absence of both a general theory and practical evidence, we are led almost inescapably to conclude that the main cause of mixed systems is almost invariably defective modelling and that prototype studies, however interesting, will always give predictions that are far removed from experimental reality. The work described in Sect. 6 suggests that the real solution is more detailed modelling, but also indicates that, in many practically important cases, this is likely to be a formidable task. There is undoubtedly a need for a parametric study of various different two-phase flow regime models to be carried out, with a view to determining whether, with suitably careful modelling, it is possible to produce hyperbolic models that arise from a rational asymptotic basis, rather than a piecemeal addition of poorly motivated equations. If such a study is to be a realistic goal, however, a great deal of information regarding the parameters and properties of two-phase flows will have to be used. The lack of availability of such information could therefore prove to be a serious matter.

There seems no prospect that the pressure exerted upon scientists and engineers to determine numerical solutions to mixed problems will decrease in magnitude. Solutions computed using methods that are theoretically unstable will therefore continue to be used to make practical predictions. The need to use large amounts of artificial numerical dissipation, unexpected code crashes, unreliable numerical convergence and occasional unbelievable predictions will all have to be tolerated if practical results are to be computed. Practical numerical tests that could inform the user when predictions were unreliable would be most helpful, but there seems little hope that numerical methods designed especially to deal with mixed problems are a realistic (or even a desirable) proposition. For mixed problems the procedure of carrying out careful and exhaustive grid sensitivity testing assumes an even greater significance than usual.

References

1. Affouf, M., Caflisch, R.E.: A numerical study of Riemann problem solutions and stability for a system of viscous conservation laws of mixed type. SIAM J. Appl. Math. 51: 605–634 (1991).
2. Bell, J.B., Trangenstein, J.A., Shubin, G.R.: Conservation laws of mixed type describing three-phase flow in porous media. SIAM J. Appl. Math. 46: 1000–1017 (1986).
3. Bick, J.H., Newell, G.F.: A continuum model for two-directional traffic flow J Appl. Math. 18: 191–204 (1961).
4. Brewster, M.E., Chapman, S.J., Fitt, A.D., Please, C.P.: Asymptotics of slow flow of very small exponent power-law shear-thinning fluids in a wedge. Eur. J. Appl. Math. 6: 559–571 (1995).
5. Drew, D.A., Cheng, L. Lahey, R.T.: The analysis of virtual mass effects in two-phase flow. Int. J Multiphase Flow 5: 233–242 (1979).
6. Drew, D.A., Wood, R.T.: Overview and taxonomy of models and methods for workshop on two-phase flow fundamentals. National Bureau of Standards Report, Gaithersburg, MD (1985).
7. Fitt, A.D.: The numerical and analytical solution of ill-posed systems of conservation laws. Appl. Math. Modelling 13: 618–631 (1989).
8. Fitt, A.D.: Mixed hyperbolic-elliptic systems in industrial problems. In: Manley, J., McKee, S., Owens, D. (eds.): Proceedings of the 3rd European Conference of Mathematics in Industry Teubner/Kluwer, Stuttgart/Dordrecht, pp. 205–214 (1990).
9. Fitt, A.D.: The character of two-phase gas/particulate flow equations. Appl. Math. Model. 17: 338–354 (1993).
10. Fletcher, C.A.J.: Computational techniques for fluid dynamics, vol. 1, 2nd edn. Springer, Berlin Heidelberg New York Tokyo (Springer series in computational physics) (1991).
11. Garabedian, P.R.: Comparison of numerical methods in transonic aerodynamics. Comput. Fluids 22: 323–326 (1993).
12. Glimm, J.: Solutions in the large for nonlinear hyperbolic systems of equations. Commun. Pure Appl. Math. 18: 95–105 (1965).
13. Gough, P.S., Zwarts, F.J.: Modelling heterogeneous two-phase reacting flow AIAA J 17: 17–25 (1979).
14. Hattori, H.: The Riemann problem for a van der Waals fluid with entropy rate admissibility criterion—isothermal case. Arch. Ration. Mech. Anal. 92: 247–263 (1986).
15. Hattori, H.: The Riemann problem for a van der Waals fluid with entropy rate admissibility criterion—nonisothermal case. J. Diff. Equations 65: 158–174 (1986).
16. Holden, H.: On the Riemann problem for a prototype of a mixed type conservation law Commun. Pure Appl. Math. 40: 229–264 (1987).
17. Holden, H., Holden, L.: On the Riemann problem for a prototype of a mixed type conservation law II. Contemp. Math. 100: 331–357 (1989).
18. Holden, H., Holden, L., Risebro, N.H.: Some qualitative properties of 2×2 systems that change type. In: Keyfitz, B.L., Shearer, M. (eds.): Nonlinear equations that change type. Springer, Berlin Heidelberg New York Tokyo (The IMA volumes in mathematics and its applications, vol. 27) (1990).
19. Hsiao, L., DeMottoni, P.: Existence and uniqueness of the Riemann problem for a nonlinear system of conservation laws of mixed type. Trans. Am. Math. Soc. 322: 121–158 (1990).
20. Igra, O., Elperin, T., Ben-Dor, G.: Blast waves in dusty gases. Proc. R. Soc. Lond. A 414: 197–219 (1987).
21. Isaacson, E., Temple, B.: The classification of solutions of quadratic Riemann problems II. MRC Technical summary report 2892, University of Wisconsin-Madison (1985).
22. James, R.D.: The propagation of phase boundaries in elastic bars. Arch. Ration. Mech. Anal. 73: 125–158 (1980).
23. Keyfitz, B.L.: Change of type in three phase flow—a simple analogue. J Diff. Equations 80: 280–305 (1989).
24. Keyfitz, B.L., Kranzer, H.C.: A system of nonstrictly hyperbolic conservation laws arising in elasticity theory. Arch. Ration. Mech. Anal. 72: 219–241 (1980).
25. Keyfitz, B.L., Kranzer, H.C.: The Riemann problem for a class of hyperbolic conservation laws exhibiting a parabolic degeneracy. J Diff. Equations 47: 35–65 (1983).
26. Liu, T-P.: The Riemann problem for general 2×2 systems of conservation laws. Trans. Am. Math. Soc. 199: 89–112 (1974).

27 Miura, H., Glass, I.I.: Development of the flow induced by a piston moving impulsively in a dusty gas. Proc. R. Soc. Lond. A 397: 295–309 (1985).
28. Mock, M.S.: Systems of conservation laws of mixed type. J. Diff. Equations 37: 70–88 (1980).
29 Mostrel, M.M.: On some numerical schemes for transonic flow problems. Math. Comput. 52: 587–613 (1989).
30. Nishida, T.: Global solution for an initial-boundary-value problem of a quasilinear hyperbolic system. Proc. Jap. Acad. 44: 642–646 (1968).
31 Pego, R.L., Serre, D.: Instabilities in Glimm's scheme for two systems of mixed type. SIAM J Numer Anal. 25: 965–988 (1988).
32. Pitman, E.B., Ni, Y.: Visco-elastic relaxation with a van der Waals type stress. Int. J Eng. Sci. 32: 327–338 (1994).
33. Prosperetti, A., Jones, A.V.: The linear stability of general two-phase flow models II. Int. J Multiphase Flow 13: 161–171 (1987).
34. Prosperetti, A., van Wijngaarden, L.: On the characteristics of the equations of motion for a bubbly flow and the related problem of critical flow. J. Eng. Math. 10: 153–162 (1976).
35 Ramshaw, J.D., Trapp, J.A.: Characteristics, stability and short-wavelength phenomena in two-phase flow equation systems. Aerojet Nuclear Company Report ANCR-1272 (1976).
36. Ransom, V.H., Hicks, D.L.: Hyperbolic two-pressure models for two-phase flow. J Comput. Phys. 53: 124–151 (1984).
37 Schaeffer, D.G., Shearer, M.: Riemann problems for nonstrictly hyperbolic 2×2 systems of conservation laws. Trans. Amer. Math. Soc. 304: 267–306 (1987).
38. Schaeffer, D.G., Shearer, M.: The classification of 2×2 systems of non-strictly hyperbolic conservation laws, with applications to oil recovery. Commun. Pure Appl. Math. 40: 141–178 (1987).
39 Schaeffer, D.G., Schecter, S., Shearer, M.: Nonstrictly hyperbolic gas laws with a parabolic line. J Diff. Equations 103: 94–126 (1993).
40. Shearer, M.: The Riemann problem for a class of conservation laws of mixed type. J Diff. Equations 46: 426–443 (1982).
41 Shearer, M.: Loss of strict hyperbolicity of the Buckley-Leverett equations for three-phase flow in a porous medium. In: Wheeler, M.M. (ed.): Numerical simulation in oil recovery, Springer, Berlin Heidelberg New York Tokyo (The IMA volumes in mathematics and its applications, vol. 11) (1988).
42. Shearer, M.: Nonuniqueness of admissible solutions of Riemann initial value problems for a system of conservation laws of mixed type. Arch. Ration. Mech. Anal. 93: 45–59 (1986).
43 Shearer, M., Schaeffer, D.G., Marchesin, D., Paes-Leme, P.L.: Solution of the Riemann problem for a prototype 2×2 system of non-strictly hyperbolic conservation laws. Arch. Ration. Mech. Anal. 97: 299–330 (1987).
44. Slemrod, M.: Admissibility criteria for propagating phase boundaries in a van der Waals fluid. Arch. Ration. Mech. Anal. 81: 301–315 (1983).
45 Smoller, J.: Shock waves and reaction-diffusion equations. Springer, Berlin Heidelberg New York Tokyo (Grundlehren der mathematischen Wissenschaften, vol. 258) (1983).
46. Smoller, J.: On the solution of the Riemann problem with general step data for an extended class of hyperbolic systems. Michigan Math. J. 16: 201–210 (1969).
47 Smoller, J.: A uniqueness theorem for Riemann problems. Arch. Ration. Mech. Anal. 33: 110–115 (1969).
48 Sod, G.A.: Numerical methods in fluid dynamics. Cambridge University Press, Cambridge (1985).
49 Stevenson, M.D., Kagan, M., Pinczewski, W.V.: Computational methods in petroleum reservoir simulation. Comput. Fluids 19: 1–19 (1991).
50. Stuhmiller, J.H.: The influence of interfacial pressure forces on the character of two-phase flow model equations. Int. J. Multiphase Flow 3: 551–560 (1977).
51 Suliciu, I.: On modelling phase transitions by means of rate-type constitutive equations. Shock wave structure. Int J Eng. Sci. 28: 829–841 (1990).
52. Trangenstein, J.A., Bell, J.B.: Mathematical structure of the black-oil model for petroleum reservoir simulation. SIAM J. Appl. Math. 49: 749–783 (1989).
53 Turolla, R., Zampieri, L., Nobili, L.: On the mathematical character of the relativistic transfer moment equations. Mon. Not. R. Astron. Soc. 272: 625–629 (1995).
54. Wang, F., Gardner, C.L., Schaeffer, D.G.: Steady-state computations of granular flow in an axisymmetric hopper. SIAM J. Appl. Math. 52: 1076–1088 (1992).
55 Weitzner, H.: Lower hybrid waves in the cold plasma model. Commun. Pure Appl. Math. 38: 919–932 (1985).

Author's address: A.D. Fitt, Faculty of Mathematical Studies, University of Southampton, Southampton SO17 1BJ, U.K.