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# Slot film cooling — the effect of separation angle

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**Summary.** Previous studies of slot film cooling have mainly concentrated on a flow geometry so arranged that the slot is flush in the containing wall. It is known that when the total pressure head of the injected flow is less than the free stream head, separation from the front of the slot is tangential to the wall (the so-called 'lid' effect). This phenomenon limits the mass flow from the slot which in turn limits the effectiveness of the cooling. An appealing strategy to enhance the mass flow is to force a non-zero angle of separation from the upstream end of the slot by geometrical means. The present study considers the influence of the geometry on mass flow and suggests possible improvements for the mean flow characteristics. Asymptotic analysis based on inviscid flow theory is used to derive a nonlinear singular integrodifferential equation (NLSIDE) for the height of the separating streamline. This equation is then solved numerically and the mass flow characteristics determined for various geometries.

## 1 Introduction

In order to increase either the life or operating temperature (and thus efficiency) of a turbine blade, a film of cooling air may be injected into the flow through small slots or holes in the surface of a turbine blade. The cooling effect is increased as the mass flow of injected air increases. Practical details of the general problem were considered in [1]

An obvious strategy for increasing the mass flow from the slot is to increase the size of an individual slot, or simply to provide more slots; evidently both of these strategies have a deleterious effect on the structural integrity of the turbine blade. An alternative proposal without this disadvantage is to increase the mass flow by a suitable choice of local geometry near the upstream edge of the injection slot. In the present study we consider geometries of the form shown schematically in Fig. 1.

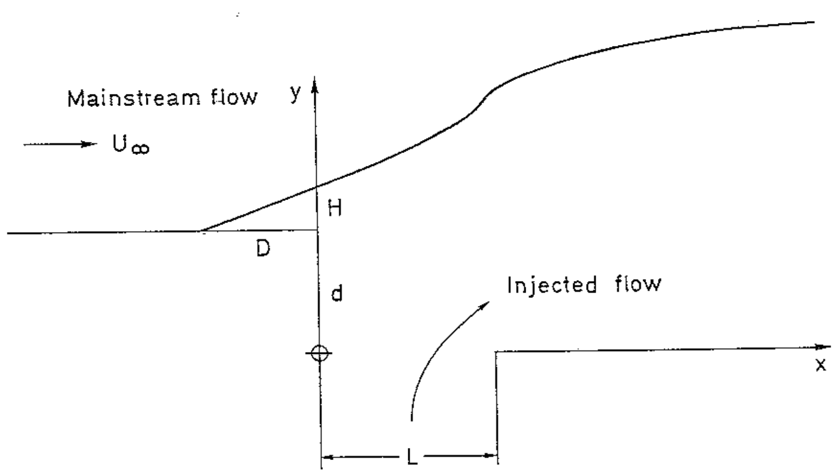


**Fig. 1.** Schematic of turbine blade slot geometry

The work contained in this paper is a generalization of that of Fitt et al. [2], and the reader is referred to that paper for details of other work in this area. For a review of earlier work in the subject, see [3]. The analogous problem for the case of suction (when the injection and free stream total pressure heads are equal) has been studied by Dewynne et al. [4] using hodograph methods. When the pressure heads are unequal however such methods are no longer applicable.

To render the problem tractable and amenable to asymptotic techniques we consider only the two-dimensional problem of injection from a slot into a free stream. The results are expected to be in qualitative agreement with three-dimensional injection through a hole.

Referring to Fig. 1, we observe a 'separation ramp' ahead of the injection slot. This has height $H$ and slope $H/D$. We also include a drop of height $d$ from the boundary upstream to that downstream. The streamline which separates the injected flow departs from this ramp. We shall investigate how the mass flow from the slot is changed by variations in $H, D$ and $d$ for a fixed value of the pressure drop between the far field and the slot.

In Section 2 we introduce the model of [2], suitably modified for the new geometry; some limiting cases of interest are also considered. In Section 3 the numerical solution of the governing NLSIDE is discussed, whilst results and conclusions are presented in Section 4.

## 2 Mathematical modelling

We now consider a mathematical model for the flow. Except for the new geometry, results and conclusions, what follows parallels the analysis of [2]. The geometry is shown in Fig. 1, the important parameters being the height $H$ and length $D$ of the separation ramp, the slot width $L$ and the drop height $d$. The model will be based upon classical inviscid thin aerofoil theory, and it is therefore assumed that the height of the separating streamline and $H$ and $d$ are small compared with $L$ and $D$. This will constrain us to examine the case when the difference between the total injection pressure and total free stream pressure is, in some sense, small. We define the small parameter $\varepsilon$ by

$$p_{ti} = p_\infty + \frac{1}{2}\varrho U_\infty^2 \varepsilon^2,$$

where $p_{ti}$ is the total pressure head of the injected fluid, $p_\infty$ is the free stream static pressure, $\varrho$ the density of the fluid and $U_\infty$ the speed of the free stream at infinity.

In the outer flow we define a potential such that the velocity is given by

$$U_\infty \boldsymbol{i} + \nabla\phi$$

We now nondimensionalize and scale by writing

$$x = L\bar{x}, \quad \phi = \varepsilon^2 L U_\infty \bar{\phi}$$

and denote the height of the separating streamline by

$$\bar{y} = \varepsilon^2 \bar{S}(\bar{x})$$

We now proceed to derive a model for the unknown $\bar{S}(\bar{x})$.

We begin by considering the free stream. Since $\bar{y} = \varepsilon^2 \bar{S}(\bar{x})$ is a streamline we must have, to leading order, that

$$\bar{\phi}_{\bar{y}} = \bar{S}', \tag{1}$$

the dash denoting differentiation with respect to $\bar{x}$ Linearising the Bernoulli equation the dimensional pressure in the outer flow is thus given by

$$p = p_\infty - \varrho U_\infty^2 \varepsilon^2 \bar{\phi}_{\bar{x}} \tag{2}$$

so that pressure variations throughout the flow are $O(\varrho U_\infty^2 \varepsilon^2)$. Since $\bar{\phi}$ satisfies Laplace's equation, its $x$ and $y$ derivatives of first order are related via the Hilbert transform. We may therefore write

$$p = p_\infty + \frac{\varrho U_\infty^2 \varepsilon^2}{\pi} \int_{-\infty}^{\infty} \frac{\bar{S}'(t)}{t - \bar{x}}\, dt, \tag{3}$$

where we have made use of the boundary condition (1).

Within the injected layer, downstream of the slot trailing edge, it is evident that the horizontal velocity must be of order $\varepsilon$ in order to produce pressure variations of the correct order of magnitude, that is $O(\varepsilon^2)$. We thus let

$$u = \varepsilon U_\infty \bar{u}.$$

From conservation of mass $v$, the vertical component of the dimensional velocity in the injected layer, is $O(\varepsilon^3 U_\infty)$, and hence

$$\bar{u}\bar{S} = M, \tag{4}$$

where $M$ is the dimensionless mass flow from the slot. Using Bernoulli's equation in the layer, we find that

$$p = -\frac{1}{2} \varrho \varepsilon^2 U_\infty^2 \bar{u}^2 + p_{ti}, \tag{5}$$

and so using Eqs. (4), (5) and the definition of $p_{ti}$, we find that

$$p = -\frac{1}{2} \varrho \varepsilon^2 U_\infty^2 \left( \frac{M^2}{\bar{S}^2} - 1 \right) + p_\infty \tag{6}$$

We also observe that, under these assumptions, the mass flow from the slot has order of magnitude $\varrho \varepsilon^3 L U_\infty$, thus the dynamic component of the total pressure within the slot is $O(\varepsilon^6 \varrho U_\infty^2)$ and hence is very much smaller than the total injection pressure. Therefore the dimensional pressure in the slot is $p_{ti}$ to lowest order

Completing the matching of the pressure across the dividing streamline, we finally arrive at the NLSIDE

$$-\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{S'(t)}{t - x}\, dt = \begin{cases} -\dfrac{1}{2} & (0 < x < 1) \\ -\dfrac{1}{2} + \dfrac{1}{2}\dfrac{M^2}{S^2} & (1 < x < \infty) \end{cases} \tag{7}$$

In the equation above, the overbars have been dropped for convenience. The relevant boundary conditions are derived by considering the height and slope of the separating streamline at the upstream end of the slot and at $x = \infty$ These conditions are

$$S(0) = H + d, \quad S'(0) = H/D, \quad S(\infty) = M, \quad S'(\infty) = 0$$

Here $H$ and $d$ have been scaled with $\varepsilon^2 L$ and $D$ with $L$. Finally, we use our knowledge of the upstream geometry to write the governing equation (first derived in the special case

$H = D = d = 0$ in [2]) as

$$\frac{H}{\pi D}\log\left(\frac{x+D}{x}\right) - \frac{1}{\pi}\int_{0+}^{\infty}\frac{S'(t)}{t-x}\,dt = \begin{cases} -\dfrac{1}{2} & (0 < x < 1) \\ -\dfrac{1}{2} + \dfrac{1}{2}\dfrac{M^2}{S^2} & (1 < x < \infty) \end{cases} \tag{8}$$

Evidently the nonlinear singular nature of this equation renders it extremely unlikely that it will be possible to find closed form solutions. It may be confirmed however that Eq. (8) has the asymptotic solutions

$$S(x) \sim H + d + \frac{Hx}{D} + o(x) \quad (x \to 0),$$

$$S(x) \sim M - \frac{M(M-d)}{\pi x} + \frac{M^2(d-M)}{\pi}\frac{\log x}{x^2} \quad (x \to \infty).$$

The aim in Section 3 of the paper will be to solve Eq. (8) numerically in order to determine the dependence of $M$ upon the parameters $H, D$ and $d$. Before mentioning two special cases of (8), it is convenient to invert this equation using standard singular integral equation theory (see, for example [5]). This gives

$$S'(x) = \frac{\sqrt{x}\,M^2}{2\pi}\int_1^{\infty}\frac{dt}{S^2(t)\sqrt{t}\,(t-x)} - \frac{H\sqrt{x}}{D\pi^2}\int_0^{\infty}\frac{\log\left(\dfrac{t+D}{t}\right)}{\sqrt{t}\,(t-x)}\,dt. \tag{9}$$

Equation (9) allows the asymptotic behaviour near to the rear edge of the slot to be easily determined. It is found that $S'(x) \sim -\log(1-x)$ as $x \to 1-$, so that at this position there is a logarithmic singularity in the slope of $S(x)$.

The coefficient of the eigenfunction which arises in the inversion has been chosen to ensure that the gradient condition at $x = 0$ $(S'(0) = H/D)$ is satisfied. Another integration may be performed on (9) and leads to

$$S(x) = \frac{M^2}{2\pi}\int_1^{\infty} S^{-2}(t)\left[-2\sqrt{\frac{x}{t}} + \log\left(\frac{\sqrt{t}+\sqrt{x}}{|\sqrt{t}-\sqrt{x}|}\right)\right]dt$$
$$- \frac{H}{D\pi^2}\int_0^{\infty}\log\left(\frac{t+D}{t}\right)\left[-2\sqrt{\frac{x}{t}} + \log\left(\frac{\sqrt{t}+\sqrt{x}}{|\sqrt{t}-\sqrt{x}|}\right)\right]dt + H + d, \tag{10}$$

where the second boundary condition at $x = 0$ has been satisfied.

### *2.1 The 'infinitesimal trip'*

An interesting limit, which we refer to as the 'infinitesimal trip' problem, exists as we let $D \to 0$ with $H = \sqrt{D}\,\bar{H}$. Thus the step height and extent both decrease to zero with the gradient of the ramp becoming infinite. Hence the slope of the separating streamline when it leaves the wall also becomes infinite. From Eq. (9) we can see that naively expanding the logarithm in the second integral term for small $D$ yields an integral that does not exist. Some care is therefore required.

However, upon setting $t = uD$ we find that the second term becomes

$$-\frac{H\sqrt{x}}{\pi^2}\int_0^\infty \frac{\log\left(1+\frac{1}{u}\right)}{\sqrt{u}\,(Du-x)}\,du$$

Clearly if $\bar{H} = O(1)$ as $D \to 0$ then (9) becomes

$$S'(x) = \frac{\sqrt{x}\,M^2}{2\pi}\int_1^\infty \frac{dt}{S^2(t)\,\sqrt{t(t-x)}} + \frac{\bar{H}}{\sqrt{x}\,\pi^2}\int_0^\infty \frac{\log\left(1+\frac{1}{u}\right)}{\sqrt{u}}\,du. \tag{11}$$

The second integral in (11) has the value $2\pi$, and relevant boundary conditions are $S(0) = d$ and $S(\infty) = M$ (the other conditions having been satisfied). It should be noted that the final term in (11) is an eigenfunction of the original integral equation (8).

In practice, the limiting case would prove difficult to realize experimentally. This is predominantly because of boundary layer effects which have been ignored. Accordingly this limiting case will not be discussed further

### 2.2 *The 'finite trip/baffle'*

Another limiting case of Eq. (8) occurs when $D \to 0$ with $H$ fixed. We refer to this as the 'finite trip' or 'baffle' problem. From (8), we see that in this limit

$$\frac{H}{\pi x} - \frac{1}{\pi}\int_{0+}^\infty \frac{S'(t)}{t-x}\,dt = \begin{cases} -\frac{1}{2} & (0 < x < 1) \\ -\frac{1}{2} + \frac{1}{2}\frac{M^2}{S^2} & (1 < x < \infty), \end{cases} \tag{12}$$

with $S(0) = H + d$ and $S(\infty) = M$ Observe that the range of the integral term in Eq. (12) extends from $0+$. In this case there is no physically viable solution. To see this, consider the simpler (linear) integral equation (valid for $x > a$)

$$\int_a^\infty \frac{S'(t)\,dt}{t-x} = \frac{H}{x} \tag{13}$$

For $a > 0$ the solution may be determined by employing the standard theory for such singular integral equations. This gives

$$S(x) = -2\tan^{-1}\sqrt{\frac{x-a}{a}} + C_1 + C_2\sqrt{x-a},$$

where $C_1$ and $C_2$ are arbitrary constants, and the condition $S(a) = 0$ may be enforced by taking $C_1$ to be zero. However, as $a \to 0+$ in the solution above, the condition $S(a) = 0$ is no longer satisfied and Eq (13) has no continuous solution. This may be contrasted with the case $a \to 0-$, when the solution is $S'(x) = \delta(x)$, so that $S$ is a Heaviside step function.

## 3 Numerical solution of the governing equation

In order to solve Eq. (8) numerically, we follow a procedure similar to that of [2], which is designed to take advantage of the form of the NLSIDE and avoid the difficulties associated with numerical differentiation and the numerical evaluation of Hilbert transforms. Using the integrated version (10) of the governing equation allows numerical calculations to be carried out more easily since the integrals are no longer singular

To discretize (10), we first assume that $S(x)$ is piecewise constant on the intervals $[t_k, t_{k+1})$ $(k = 1, \ldots, N-1)$ and, taking $t_1 = 1$, write

$$S(x_i) = \frac{M^2}{2\pi} \sum_{k=1}^{N-1} S^{-2}(t_k) \int_{t_k}^{t_{k+1}} \left( -2\sqrt{\frac{x_i}{t}} + \log\left( \frac{\sqrt{t} + \sqrt{x_i}}{|\sqrt{t} - \sqrt{x_i}|} \right) \right) dt + E_{N-1}(x_i) + P(x_i, D, H, d) \quad (i = 1, \ldots, N), \tag{14}$$

where

$$E_{N-1}(x_i) = \frac{M^2}{2\pi} \int_{t_N}^{\infty} S^{-2}(t) \left( -2\sqrt{\frac{x_i}{t}} + \log\left( \frac{\sqrt{t} + \sqrt{x_i}}{|\sqrt{t} - \sqrt{x_i}|} \right) \right) dt$$

and

$$P(x, D, H, d) = -\frac{H}{D\pi^2} \int_0^{\infty} \log\left( \frac{t + D}{t} \right) \left( -2\sqrt{\frac{x}{t}} + \log\left( \frac{\sqrt{t} + \sqrt{x}}{|\sqrt{t} - \sqrt{x}|} \right) \right) dt + H + d.$$

Evaluating the integrals analytically wherever possible and estimating the error term $E_{N-1}$ for large enough $t_N$, we derive the iterative scheme

$$S_{j+1}(x_i) = \sum_{k=1}^{N-1} S_j^{-2}(t_k)\, A_{ik} + \frac{M^2}{\pi} x_i S_j^{-2}(t_N)\, Q\left( \sqrt{\frac{t_N}{x_i}} \right) + P(x_i, D, H, d) \quad (i = 1, \ldots, N, \quad j = 1, 2, \ldots), \tag{15}$$

where

$$A_{ik} = \frac{M^2}{2\pi} \left( 2\sqrt{x_i} \left( \sqrt{t_k} - \sqrt{t_{k+1}} \right) + (t_{k+1} - x_i) \log\left( \frac{\sqrt{t_{k+1}} + \sqrt{x_i}}{|\sqrt{t_{k+1}} - \sqrt{x_i}|} \right) - (t_k - x_i) \log\left( \frac{\sqrt{t_k} + \sqrt{x_i}}{|\sqrt{t_k} - \sqrt{x_i}|} \right) \right),$$

$$Q(\alpha) = \alpha + \frac{1}{2}(1 - \alpha^2) \log\left( \frac{\alpha + 1}{\alpha - 1} \right)$$

The integral arising in the definition for $P$ cannot be calculated in closed form, but may be evaluated numerically. For the computations described below, the highly accurate and efficient NAG routine D01AJF was used.

This scheme may be used to determine the solution for $1 \leqq x \leqq \infty$, whereupon the solution in the region $[0, 1)$ may be calculated from Eq. (14). The essential nonlinearity of the equation makes it extremely unlikely that convergence of the numerical method can be proved, and we do not address this question here. In practice it is found that in order to ensure convergence from any (positive) initial guess for $T_0(x)$, some relaxation must be employed. Accordingly we supplement the scheme (15) with

$$T_{j+1}^{\text{final}}(x_i) = T_j(x_i) + \theta\big(T_{j+1}(x_i) - T_j(x_i)\big),$$

where $\theta$, the relaxation parameter, is less than unity. It should also be noted that, for given $D$, $H$ and $d$, the mass flow $M$ is unknown at the outset. It may be determined however by using the condition that $S(\infty) = M$ For a fixed geometry, an initial value of $M$ is guessed, and the problem is solved. Depending on the value of $S(\infty)$, the estimate for $M$ is altered until finally the prescribed $M$ coincides with $S(\infty)$. A practical difficulty concerns the accurate estimation of a value for $S(\infty)$ when a necessarily finite-length mesh is employed. From the asymptotic estimates we know that

$$S(x) \sim S(\infty) - \frac{K}{x} + O\left(\frac{\log x}{x^2}\right),$$

where $K$ is a strictly positive constant, so that the approach to the limiting value $M$ is not particularly rapid and it may be expected that the mesh will have to extend a comparitively large distance if accurate results are to be produced. Bearing in mind however the fact noted earlier that the slope of $S(x)$ has a logarithmic singularity at $x = 1$, it is clear that a fairly fine mesh will be required near the rear of the slot. The most practical answer to these twin requirements is to use a non-uniform mesh. The situation is comparable with the calculation of turbulent boundary layers (see, for example [6]), and for the calculations reported below a mesh defined by

$$x_{k+1} = 1 + dx_0\left(\frac{1-g^k}{1-g}\right) \qquad (k = 0, \ldots, N-1)$$

was used. Here $dx_0$ is the initial mesh spacing and $g$ the mesh scale factor.

Some discussion of the iterative process required to determine $M$ for a given geometry is relevant. For the case when $H = 0$, so that there is merely a step down from the front to the rear of the slot, the analysis is similar to that in [2], and $M$ may be removed from the problem by a simple rescaling of $S(x)$, thus obviating any iterative determination of $M$. In principle $M$ may also be eliminated from Eq. (10); multiplication of Eq. (8) by $S'(x)$ and integration between 0 and $\infty$ cancels out the middle term of the equations and leads to

$$\frac{H}{\pi D}\int_0^\infty S'(x)\log\left(\frac{x+D}{x}\right)dx = -M + \frac{M^2}{2S(1)}, \tag{16}$$

which in principle allows $M$ to be determined in terms of $S(x)$ and $D$ only. The appearance of the derivative in Eq. (16) renders the integral difficult to compute however. Even this problem may be avoided; if Eq. (9) is used in Eq. (16) an expression analogous to Eq. (10) but not involving $M$ or derivatives of $S(x)$ may be derived. Some experiments have shown, however, that the complexity of the expression for numerical purposes renders the iterative method for determining $M$ more efficient.

## 4 Results and conclusions

In Figs. 2, 3, and 4 numerical results are displayed with $M$ plotted against $D$, $H$ and $d$ respectively. Figure 2 shows the mass flow $M$ vs. the step extent $D$ for $H = 0.1$ and $d = 0$. Clearly as $D \to \infty$ the problem is identical to the case with $H = 0$ and $d = 1/10$, so that $M$ is bounded below.

In Fig. 3 $M$ is shown as a function of $H$ for $D = 1$ and $d = 0$. For values of $H$ in the range 0 to 1/2 this relationship is approximately linear. Similarly, Fig. 4 shows the dependence of $M$ on $d$ (here $H = 0.01$ and $D = 1.0$) for values of $d$ lying between 0 and 3/2.

Finally, in Fig. 5 representative separation streamlines are shown for the parameters $D = d = 1/2$ and $H = 0, 0.1$ and $0.2$ in which case the respective dimensionless mass flows are $M = 1.70, 2.16$ and $2.67$. The logarithmic singularity in $S'(x)$ at $x = 1$ may be clearly observed.

In conclusion, the current study has extended the theory of [2] to cover new, more general

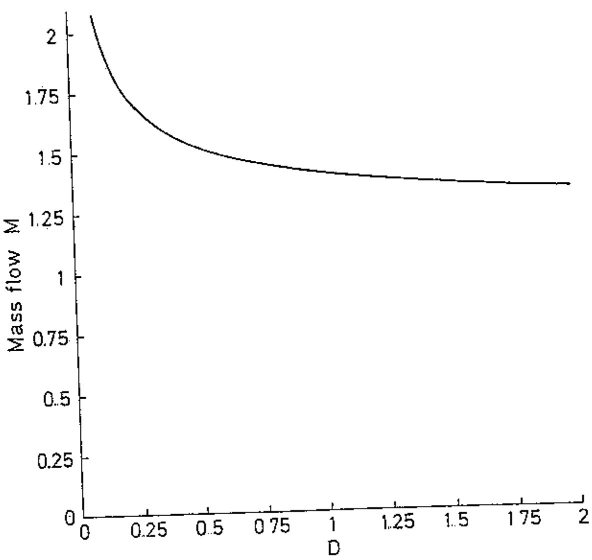


**Fig. 2.** Mass flow $M$ plotted as a function of $D$ for $d = 0$, $H = 1/10$

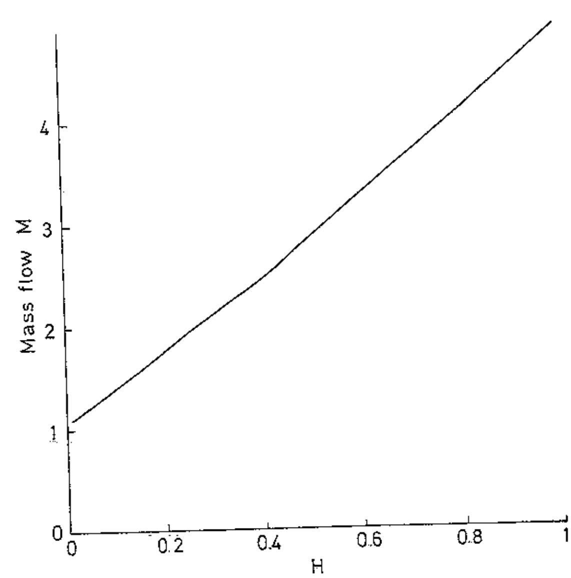


**Fig. 3.** Mass flow $M$ plotted as a function of $H$ for $d = 0$, $D = 1$

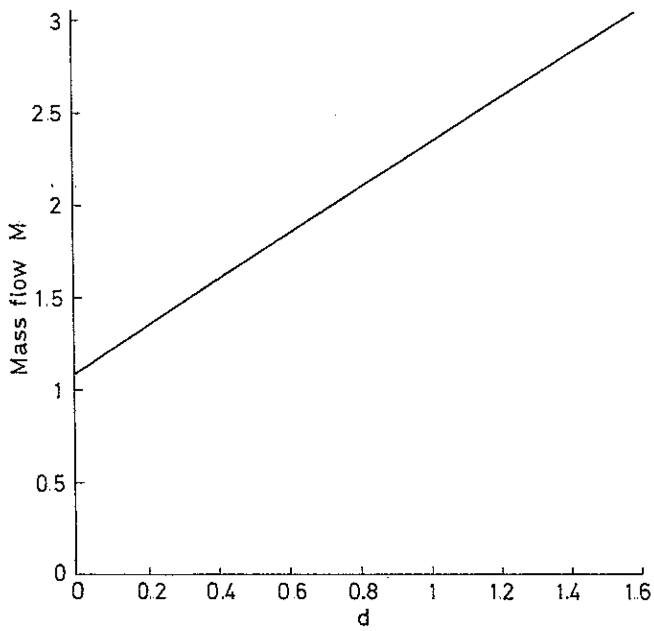


**Fig. 4.** Mass flow $M$ plotted as a function of $d$ for $H = 1/100, D = 1$

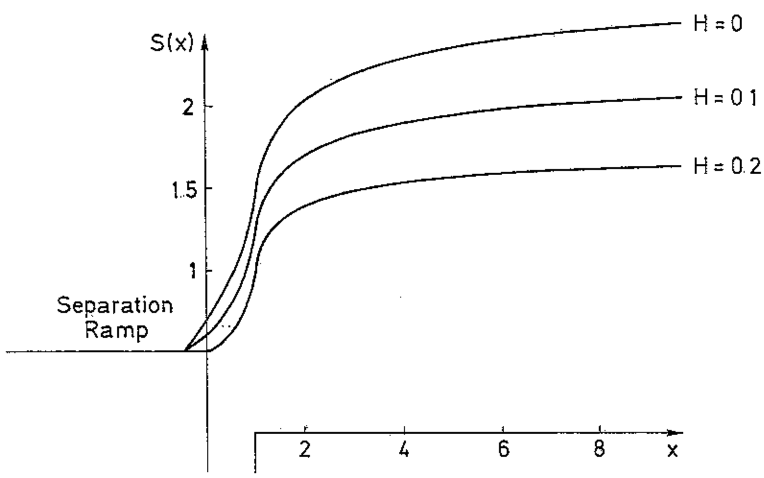


**Fig. 5.** Shape of dividing streamline $S(x)$ for various values of $H$ when $D = d = 0.5$

geometries that are likely to be of more significant practical use in the field of thermal protection of turbine blades.

One practical problem encountered in turbine blade cooling is to maintain the structural integrity of the blade whilst injecting a sufficient quantity of cool gas to provide the necessary amount of protection; slots that are either too large or too numerous may lead to an unacceptable weakening of the blade. With the previous model described in [2] the only way to increase mass flow was to increase the slot width. In this model, extra freedom is available — the parameters $H$, $D$ and $d$ — to ensure an increase in the mass flow without resorting to increased slot size. The question of how to fix $H$, $D$, $d$ and the slot width $L$ uniquely and optimally will depend upon the mechanical and other properties of the blade.

The current study also suggests the possibility of extending the analysis to three-dimensional flow: the case of injection through a hole.

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