TWO-PHASE FLOW METHODOLOGY AND MODELLING

A.D. Fitt, Faculty of Mathematics, Southampton University, Hampshire, England

1. Introduction

The problem of how to treat flows in which there is more than one phase present has long been a difficult one. Recently interest has increased in such flow problems, notably in the nuclear power and defence industries where multiphase mixtures are commonplace. By 'multiphase' here we mean flows where the multiconnectedness or complexity and unsteadiness of the interphase boundaries makes phase 'tracking' impractical.

In the past a popular way of proceeding has been to average the equations in some way, introducing some form of void fraction function which indicates how much of each phase is present. Unfortunately the progress from generalized single - component equations to 'two - phase' equations has proved to be littered with problems (witness the controversy in [6]). These difficulties frequently arise from insufficient generality of the basic equations of motion, failures in the averaging procedure, or a neglect of important quantities. Very often in the past the result has been a set of conservation laws which possesses complex characteristics. The consequences of this are severe. The Cauchy problem becomes ill-posed, and any attempt at a numerical solution to such sets of equations is likely to suffer from exponential instability and other problems. Additionally, some of the best methods for shock capturing (for example the method of random choice [2]) and other robust implicit schemes (for example the implicit MacCormack method as described in [7]) cannot now be used at all, as they all rely on the existence of real characteristics. From many angles therefore the appearance of complex characteristics is highly undesirable.

Solutions to these shortcomings of the equations have varied in their ingenuity and physical reality, ranging from the completely artificial where terms in the equations giving rise to complex characteristics are ignored) to the more considered but still dubious piecemeal addition of extra terms to certain equations. What is really required however is a more general approach to the modelling of the problem, carried out in such a way that the assumptions which have been made can be clearly appreciated and justified.

In some ways the 'moral' of the present study [1] is a pessimistic one. The theory tells us that two - phase flow is essentially a MODELLING problem, and the modelling will change for different flows. It is too much to hope that a single set of simple equations will suffice for all two phase flows. Instead we must proceed from a GENERAL set of equations, and use modelling skills, combined with suitable small parameter analysis to indicate which terms should be included in the final equations of motion which we wish to solve to determine the flow. If we are unable to accomplish this, then the fault is with the modelling (for example, insufficient knowledge of turbulent effects) and not the basic equations themselves. It is also worth stating that most of the initial development of the equations of motion is very similar to that carried out in [3]. Both ensemble and cross-sectional averaging procedures are used before the gas/particulate flow regime is examined.

[1] A fuller version of this paper will be submitted to Int. J. Multiphase Flow

Hj. Wacker and W. Zulehner (eds.),
Proceedings of the Fourth European Conference on Mathematics in Industry, 277–281.

2. General Equations for Two-Phase Flow and their One-Dimensional Interpretation

Because of the anticipated sensitivity of the modelling, we begin consideration of two-phase flow equations from a suitably general viewpoint. This entails proposing equations for each phase separately, before averaging. Using standard theory, the equations of motion and associated jump conditions for each phase may be written

$$(1) \qquad \frac{\partial}{\partial t}(\rho\Sigma) + \nabla.(\rho \boldsymbol{q}\Sigma - \boldsymbol{J}) = \rho f, \quad f_i = [(\rho\Sigma(\boldsymbol{q} - \boldsymbol{q}_i) + \boldsymbol{Q})\ \hat{\boldsymbol{n}}_i]_1^2.$$

Here ρ represents the phase density, $\boldsymbol{q}_i$ the interfacial velocity, f_i the interfacial source density term and $\hat{\boldsymbol{n}}_i$ the unit interface normal. The divergence term is to be suitably interpreted depending on whether its operand is a vector or a scalar. We then develop the equations of motion for each phase by suitably choosing the variables in (1).

Accepting now that to use the above equations separately for each phase would involve an unmanagable amount of interface tracking, we must now consider how to average the equations. First, we employ a (fully three-dimensional) standard ensemble averaging process where the average of a quantity f is given by

$$\bar{f} = \int_M f(\boldsymbol{x}, t; \omega) dm(\omega)$$

where M is the total probability space and the measure $m(\omega)$ represents the probability of observing state ω. Since more than one phase is present, we cannot assume that the derivative of $\bar{f}$ will be equal to the average of the derivative of f. Instead, phase indicator functions must be introduced (see [3]). Using these, and various other properties of the averaging operator, it is possible to propose general three-dimensional averaged equations. It is also possible to include the effects of fluctuations such as turbulence, but for the purposes of the present study these are ignored. At this juncture, we could start to specialize to the particular regime of gas/particulate flow, but first we simplify further by employing quasi one-dimensional assumptions and area averaging. For any variable $\bar{f}$ which has already been ensemble averaged we define

$$< \bar{f}(\boldsymbol{x}, t) > = \frac{1}{A(x)} \int\int_{A(x)} \bar{f}(\boldsymbol{x}, t) dA$$

where $A(x)$ represents the cross - sectional area of the channel at ordinate x. Again, standard averaging results may be used (see, for example [4]), and the cross-sectional average of the general conservation law becomes

$$< \bar{\Sigma}\bar{\alpha}_k\bar{\rho}_k >_t + \frac{1}{A}\oint_C \frac{\bar{\alpha}_k\bar{\rho}_k\bar{\boldsymbol{q}}_k(\bar{\Sigma}.\boldsymbol{n})}{N} ds + \frac{1}{A}[A < \bar{\Sigma}\bar{\alpha}_k\bar{\rho}_k\bar{\boldsymbol{q}}_k >]_x . \boldsymbol{e}_x =$$

$$\frac{1}{A}\oint_C \frac{\bar{\alpha}_k\bar{\boldsymbol{J}}_k.\boldsymbol{n}}{N} ds + \frac{1}{A}[A < \bar{\alpha}_k\bar{\boldsymbol{J}}_k >]_x.\boldsymbol{e}_x + < \bar{f}_k > + < \bar{j}_k > + < \bar{\Sigma}_{ki}\bar{\Gamma}_k >$$

Now choosing $\Sigma, \boldsymbol{J}$ and f as usual and resolving the momentum equation in the axial direction gives six equations of motion, three for each phase.

3. The Modelling of Gas/Particulate Two-Phase Flows

We must now consider how to model specific terms in the equations of motion. This modelling must reflect the exact nature of the flow which we are considering; if we do not make the correct assumptions we cannot expect to end up with a hyperbolic system.

Firstly, we assume that $A(x)$ = constant and neglect all purely algebraic source terms since these cannot affect the hyperbolicity of the system. Next, we assume that phase 1 is a (compressible) gas phase, and phase 2 an (incompressible) solid phase which is dispersed in the form of 'blobs' within the continuous phase 1. The gas phase is assumed to be a Newtonian viscous fluid, and the particles of the solid phase are regarded as being composed of highly viscous fluid, but with no displacements inside any parts of phase 2 so that the stress tensor has only pressure components Further, the particles are assumed to be so dispersed that there can be no stress created between neighbouring solid particles, and collisions are neglected. Then *within each solid particle* the pressure p_2 is constant, and depends on the pressure in phase 1. To model the interfacial momentum and work terms it is normal to separate out the pure interfacial terms from the mean flow terms. This introduces the interfacial pressures p_{1i} and p_{2i} which we assume are related to the bulk pressures p_1 and p_2 via some inviscid flow calculation. The incompressibility of the solid phase requires that $p_2 = p_{1i} = p_{2i}$, and to relate p_1 to these pressures we use the fact that for inviscid flow streaming with velocity V past a sphere of radius a it is easily shown (see, for example [7]) that the pressure on the surface of the sphere will have the form

$$p\,|_{r=a} = p_\infty - \frac{1}{2}\rho_\infty V^2 F(\theta)$$

where p_∞ and ρ_∞ are the pressure and density respectively far away from the sphere and θ is the azimuthal angle. This leads to the introduction of an 'interfacial pressure coefficient' C_s which remains when all the θ-dependence has been averaged out

The bulk added interfacial terms, representing many of the forces which have been averaged out, must also be considered. A careful non-dimensionalization is needed here in general to decide which of these terms must be retained. In the present study, only the virtual mass and drag terms are important. Following [1] we use a standard virtual mass term and assume that the drag is proportional to the square of the relative velocity The jump conditions allow the interfacial terms relevant to one phase to be inferred from the other.

It remains to consider the gas phase energy equation Assuming no heat flux and a standard relation between the interfacial work and interfacial momentum transfer, and closing the system with the standard perfect gas law, the final 'working equations' are

$$(\alpha_1\rho_1)_t + (\alpha_1\rho_1 u_1)_x = 0$$

$$(\alpha_2)_t + (\alpha_2 u_2)_x = 0$$

$$(\alpha_1\rho_1 u_1)_t + (\alpha_1\rho_1 u_1^2 C_{u1})_x + \alpha_1 p_{1x} = -C_s\rho_1(u_1 - u_2)^2\alpha_{1x} + C_{vm}\alpha_2\rho_1[(u_{1t} + u_1 u_{1x}) - (u_{2t} + u_2 u_{2x})]$$

$$(\alpha_2\rho_2 u_2)_t + (\alpha_2\rho_2 u_2^2 C_{u2})_x + \alpha_2 p_{1x} = \alpha_2[C_s\rho_1(u_1 - u_2)^2]_x - C_{vm}\alpha_2\rho_1[(u_{1t} + u_1 u_{1x}) - (u_{2t} + u_2 u_{2x})]$$

$$(\alpha_1\rho_1(e_1 + u_1^2/2))_t + (\alpha_1\rho_1 u_1 C_{e1}(e_1 + u_1^2/2))_x + (p_1 u_1 \alpha_1)_x + p_1\alpha_{1t} = C_s\rho_1(u_1 - u_2)^2\alpha_{1t} + u_2 C_{vm}\alpha_2\rho_1[(u_{1t} + u_1 u_{1x}) - (u_{2t} + u_2 u_{2x})]$$

Here subscripts refer to phase number or differentiation, α is the void fraction, ($\alpha_1 + \alpha_2 = 1$), ρ represents density, u velocity, p pressure, e is internal energy, C_s and C_{vm} are respectively the drag and virtual mass coefficients, and C_{u1}, C_{u2} and C_{e1} are profile parameters in the momentum and energy equations.

4. Hyperbolicity Analysis of the Equations

We must now analyze the hyperbolicity of the working model. It should be emphasized that in order to accomplish this task in any reasonable length of time, a symbolic manipulator (MAPLE was used in the calculations reported below) is essential. Indeed, it is fair to say that before the advent of such algebraic computation systems the calculations below would have been well-nigh impossible to perform. Before dealing specifically with the gas/particulate flow case, we note a number of other special cases.

4.1 Case 1: Constant-pressure model, no added terms

The model with $p_1 = p_2 = p_{1i} = p_{2i}, C_{u1} = C_{u2} = C_{e1} = 1$ and $C_{vm} = C_s = 0$ has long been used for the modelling of many different two-phase flows. The hyperbolicity result here (see [5]) is that the eigenvalues are $\lambda = u_1$ and $\lambda = yc + u_1$ where y is a root of the quartic equation

$$y^4 - 2Vy^3 + y^2(V^2 - 1 - q) + 2Vy - V^2 = 0$$

and $c^2 = \gamma p/\rho, V = (u_2 - u_1)/c, q = \alpha_2\rho_1/(\alpha_1\rho_2)$. Four real roots exist if and only if $V^2 > (1 + q^{\frac{1}{3}})^3$. Hence the system is totally hyperbolic only for large enough relative speeds.

4.2 Case 2: Bubbly Flow

One model for bubbly flow, where we assume that phase 1 is an inviscid fluid, and phase 2 is composed of incompressible gas bubbles is similar to the working system, but requires no energy equation. With $\rho_1 =$ constant, we find that there are two zero eigenvalues, and the other two satisfy a quadratic equation. We can analyze this by setting $\epsilon = \rho_2/\rho_1 \ll 1$. Then the leading order quadratic exhibits no dependence on C_{u2}. For typical values of say $C_{vm} = 1/2, C_{u1} = 1$, we find that the condition for real roots is

$$-4\alpha_1 C_s^2 + 4\alpha_1^3 C_s - 6\alpha_1 C_s + 2C_s + 2\alpha_1^2 - 3\alpha_1 + 1 < 0,$$

so that for $C_s = 0$ the roots are real so long as $1/2 < \alpha_1 < 1$. For non-zero values of C_s, the range of values which α_1 may take INCREASES. In other words the *interfacial pressure term helps the hyperbolicity*. Physically, it seems very reasonable that the modelling should set a limit on the size of α_2.

4.3 Case 3: Gas/Particulate flow with added terms

Now we consider the working equations. A MAPLE calculation for the full system yields a very large determinant with nearly a thousand terms. Suppose however we decide to ignore the effects of profile parameters for the present and set $C_{u1} = C_{u2} = C_{e1} = 1$. The main difference between the present case and the bubbly flow considered above is that here the density ratio is large, as we assume that the density of the solid is much greater than that of the gas. setting $\epsilon = \rho_1/\rho_2 \ll 1$, we find that to first order the eigenvalues are given by $\lambda = u_1, u_2, u_2$ and the roots of a quadratic with discriminant

$$C_{vm}^2(\gamma - 1)^2(u_1 - u_2)^2\alpha_2^2 + 4c^2\alpha_1(\alpha_1 C_{vm} + \alpha_1 - C_{vm}).$$

Hence the conclusions here are totally different from bubbly flow: As usual the hyperbolicity of the system is guaranteed for large enough relative velocities, but otherwise we have hyperbolicity only when

$$\alpha_1 > \frac{C_{vm}}{1 + C_{vm}}$$

Thus in gas/particulate flow, the interfacial pressure term does not help the situation: C_s does not appear to lowest order. Moreover, the extra virtual mass term can have a *bad* effect on the hyperbolicity, since for $C_{vm} = 0$ the system is hyperbolic, but for $C_{vm} = 1/2$ for example we require $\alpha_1 > 1/3$. Clearly further analysis is needed of the gas/particulate flow case. One way of proceeding is to note that for all cases considered so far, the biggest threat to hyperbolicity seems to arise for low relative velocities. Accordingly we analyse the case where $u_1 = u_2$ and $\epsilon = r$ say is not nessecarily small. Now the eigenvalues are given by $\lambda = u_1$ (three times) and the roots of a quadratic with discriminant

$$4\alpha_1 c^2(\alpha_1(rC_{vm} - C_{vm} - 1) + C_{vm})(\alpha_1^2(r-1) - r\alpha_1 + rC_{vm})$$

Certainly for $C_{vm} = 0$ the system is hyperbolic, but for $C_{vm} = 1/2$ for example (a reasonable value) the hyperbolicity requirement (for $r < 2/3$) is

$$\alpha_1 > \frac{1}{3 - r}.$$

5. Conclusions and Possibilities for Further Work

In this short study there has only been time to indicate how to proceed and to analyze some simple cases. The conclusion for gas/particulate flow is that more terms are needed in the model Both profile parameters and fluctuation terms have been totally ignored. Also, for internal ballistics flows of the type which we ultimately wish to study, interparticulate stresses and collisions may be important Clearly, much work remains

References

[1] Batchelor, G K. (1979) 'An Introduction to Fluid Dynamics' C U P.

[2] Chorin, A. (1976) 'Random Choice Solutions of Hyperbolic Systems' J Comput Phys **22** pp 517-536

[3] Drew, D A. & Wood, R.T. (1985) 'Overview and Taxonomy of Models and Methods for Workshop on Two-Phase Flow Fundamentals' National Bureau of Standards, Gaithersburg, Maryland

[4] Fitt, A.D (1987) 'A Rigorous Derivation of a Set of Two-Dimensional Equation for Gas/Particulate Flow in a Channel' CIT Maths and Ballistics Report No. MB 1/87

[5] Fitt, A D (1988) 'Mixed Hyperbolic-Elliptic Systems in Industrial Problems' Proc ECMI88 Strathclyde, Teubner

[6] Gough, P.S & Zwarts, F J (1979) 'Modeling Heterogeneous Two-Phase Reacting Flow' AIAA J **17** pp 17-25

[7] MacCormack, R.W. (1982) 'A Numerical Method for Solving the Equations of Compressible Viscous Flow' AIAA J **20** pp 1275-1281